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A NEW METHOD FOR TREATING OPTIMAL TRAJECTORIES WITH RESTRICTED SEGMENTS

by Thomas L. Vincent and Joseph D. Mason

Prepared by
THE UNIVERSITY OF ARIZONA
Tucson, Ariz.
for

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NOMENCLATURE

a	constant, a particular control variable (p. 29)
A	wing area of aircraft
b	constant
C	constant
C_D	drag coefficient
d	constant
D	drag
e	eccentricity (p. 51)
E	a particular state variable (p.29)
f	integer
F	functional form defined on p. 3
g	acceleration due to gravity
G	universal gravitational constant, functional form defined on p. 51
h	angular momentum (p. 51)
H	Hamiltonian function defined on p. 5
k	constant of proportionality (p. 42)
K	constant thrust
ℓ	a real positive number
L	switching function defined on p. 46
m	integer, mass
M	mass of primary celestial body
n	integer
p	parameter
r	(thrust minus drag)/weight, radius from primary celestial body (p. 48)
R	ratio r_3/r_2
s	integer
S	functional form defined on p. 29
t	time
T	thrust
u	control variable, non-dimensional velocity (p. 15)
v	integer, velocity
v_r	reference velocity defined on p. 15
V	ratio v_3/v_2
W	weight of aircraft
x	range
y	state variable, altitude
Z	functional form defined on p. 3
Z*	functional form defined on p. 5
α	functional form defined on p. 9
β	functional form defined on p. 9, mass flow rate (p. 40)
γ	flight path angle
η	non-dimensional altitude (p. 15)
θ	functional form defined on p. 10
λ	Lagrange multiplier, function defined on p. 54
μ	Lagrange multiplier
ν	Lagrange multiplier
ξ	non-dimensional range (p. 15)

π Lagrange multiplier

ρ_o reference density

τ non-dimensional time

ϕ functional form defined on p. 3,
polar angle (p. 48)

Φ functional form defined on p. 8

χ steering angle (p. 39), (p. 48)

ψ functional form defined on p. 3, 10

Subscripts

C constant

e $1...r \leq (n+1)f$

f final point

i 1...n

j 0...s

k 1...m

NO non-optimal control

o optimal

q 1...f

1,2... initial and intermediate points

Notation

A dot denotes differentiation with respect to the independent variable time.

A prime denotes differentiation with respect to non-dimensional time.

A "+" sign refers to conditions just after a corner.

A "-" sign refers to conditions just before a corner.

ABSTRACT

A new procedure for handling optimization problems with corners resulting from the imposition of restrictions is presented in this paper. The requirements which must be met in order to apply the methods presented here are that the dynamical equations of constraint contain only one control variable and that these equations can be analytically integrated along any segment of the trajectory in which the control law is restricted. Under such circumstances it is shown that the restricted segments of the optimal trajectory may be effectively eliminated in the process of determining the changes in the Lagrange multipliers, state variables, and Hamiltonian function from one side of the restricted segment to the other. A set of conditions which may be used to determine the changes or "jumps" in the Lagrange multipliers and Hamiltonian are obtained directly from a set of corner conditions developed here for trajectories with discontinuities in the state variables. With the use of these conditions, there is no need to integrate the Euler equations along the restricted arc as is usually done in the literature.

It is shown that the discontinuous solution resulting from the elimination of the restricted segment may be made continuous again, if desired, through the use of parameters.

A number of example problems using the theory developed in the paper are presented to show the wide range of applicability of the theory. The examples include optimization problems with both first order and second order state variable constraints, optimal staging of a rocket vehicle with coasting periods, and optimal orbit transfer with coasting periods.

SECTION I

INTRODUCTION

An optimal trajectory between any two given points in state space, frequently will not be composed of one smooth continuous arc. Often, as has been demonstrated in the literature, an optimal trajectory is composed of a number of arcs which are joined at "corner points". In this paper, a corner point is defined to be any point of the optimal trajectory at which the optimal control variable or its rate of change is discontinuous or a point at which the optimal control law changes form.

Constraints on state and/or control variables in a particular optimization problem will generally lead to solutions in which a portion of the optimal trajectory is composed of a state variable or control variable boundary. In this case, the corner points correspond to the entrance and exit points to and from the boundary. The optimal trajectory may or may not be smooth at these points.

The requirement that an optimal trajectory have certain prescribed discontinuities in the state variables or follow certain modes of operation at some point or points along the trajectory will also lead to solutions with corner points. In these cases the corner points will correspond to the points of discontinuity in the state variables or the points in which a change is made from an optimal to a prescribed mode of operation. Here again, the resulting trajectory will not necessarily be smooth at these points.

There are numerous other situations in optimal control synthesis in which the resulting trajectory will contain a number of corners.

The type of corner points of primary interest in this paper are those which result either from the imposition of state variable constraints, from the possibility of discontinuities in state variables, or from certain allowable modes of control variable operation. An example of each of these situations may be found in the flight of a multi-stage rocket vehicle. Due to structural considerations, limitations must be imposed on certain combinations of the state variables and, since the vehicle is multi-staged, the mass may be discontinuous at various points along the trajectory. Finally, since staging takes place during a coast, this coasting period represents a mode of operation during which certain control variables are known.

A fundamental step in solving problems of this nature is to determine how to optimally join the arcs of this trajectory together at the corner points. Methods for handling corners resulting from state variable constraints are to be found in the literature. These methods are based on the concept of introducing the equation of constraint into the formulation of the problem with the use of Lagrange multipliers. As a result of this procedure, two sets of Euler equations are obtained. One set is applicable for the unrestricted segments of the trajectory, and one set is applicable to the trajectory which follows the constraint. In addition, corner conditions are obtained which govern the joining of these segments. The difficulty associated with this procedure is that the jumps in the Lagrange multipliers and the Hamiltonian function are obtained separately at the entrance and exit points. Thus the Euler equations must be integrated along the state variable constraint in order to determine the proper values for the Lagrange multipliers when leaving the constraint. If the bounded portion of the trajectory is long and if the Euler equations applicable to this portion of the trajectory cannot be integrated analytically (they usually can't), then considerable errors and difficulties can be introduced into the solution by the numerical integration of these equations.

In this paper it is demonstrated that for problems with one control variable, the conditions needed for optimally joining corner points can be obtained without integrating the Euler equations along the restricted trajectory. This may be done by confining attention to the unrestricted portion of the trajectory and considering the optimal trajectory to be discontinuous between points which contain the restricted trajectory. If this procedure is followed, a set of "jump" conditions in terms of entering and leaving points are obtained for the state variables by analytically integrating the dynamical constraint equations subject to the restriction.* The "jump" conditions for the associated Lagrange multipliers are obtained immediately from the corner conditions which are developed here for trajectories with discontinuities in the state variables. This gives directly the change in the state variables, Lagrange multipliers, and Hamiltonian function between the points where the optimal path meets and leaves the bounding constraint. This means that the only time the Euler equations need be integrated is over portions of the trajectory which are unrestricted. It is further demonstrated in this paper that a problem with a discontinuity resulting from considerations just discussed may be transformed into a problem with continuous state variables through the use of parameters. Although the two cases are theoretically equivalent, this procedure may be useful in working out the solutions to a specific problem.

A number of examples which include the use of parameters are worked out in detail in this paper to illustrate the new procedure

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*The requirement of being able to analytically integrate the restricted equations of motion is met for a large class of important problems as will be shown in Section IV.

which may be used to handle problems formulated with state variable constraints, coasting arcs, and discontinuities in state variables. One of these examples of particular interest which involves coasting arcs and discontinuities is the determination of the optimal flight of a multistage rocket vehicle.

The methods for handling restricted arcs presented in Section III involve a discussion of the use of parameters. Since the necessary optimizing conditions for variational problems with parameters is not readily found in the literature, the development of the necessary optimizing conditions for a general problem of Bolza formulated with parameters is presented first in Section II. Also included in this section is the introduction of the functional z* with an explanation of how the requirement dz* = 0 results in a necessary end point corner condition for an extremum. This condition is fundamental to the methods presented in Section III.

SECTION II

AN EXTENSION OF THE PROBLEM OF BOLZA

The Formulation

<u>The introduction of parameters and additional constraints</u> - The problem of Bolza, as formulated by Bliss[1], may be written in terms of the modern concepts of state and control variables. Using this modern notation, the problem of Bolza is that of finding in a class of state and control variables functions

$$y_i(t) \quad u_k(t) \quad i = 1,\dots,,n \quad k = 1,\dots,m \tag{2.1}$$

satisfying differential equations and end conditions of the form

$$\dot{y}_i = \Phi_i(y_1,\dots,y_n,u_1,\dots,u_m,t), \tag{2.2}$$

$$\Psi_e(t_1,t_f,y_{i1},y_{if}) = 0, \quad e = 1,\dots,v \leq 2n + 2 \tag{2.3}$$

those which minimize a sum of the form

$$Z(t_1,t_f,y_{i1},y_{if}) + \int_{t_1}^{t_f} F(y_i,u_k,t)dt. \tag{2.4}$$

In this section the problem of Bolza will be extended so that parameters and additional constraints relating both end points and corner points can also be included in the above formulation. The optimizing conditions which result from this extension will be needed for the development of the methods for handling restricted arcs as presented in Section III.

In order to formulate the problem of Bolza to include parameters, two approaches may be used to introduce them. They may be introduced either as a separate quantity resulting in three types of variables, namely, state, control, and parameter, or the parameters may be considered as state variables with an equation of constraint given by, $\dot{p} = 0$.

The former approach has been used by Pontryagin[2] to introduce parameters into the formulation of the Maximum Principle. The latter approach is simpler, however, and will be used here. Even though a parameter will be treated as a state variable in the following analysis, it will be given a special symbol, p, in order to avoid confusion.

Constraints relating to end points and/or corners will be introduced by employing the following notation for these conditions

$$\Psi_e(t_q, y_{iq}, p_j) = 0 \quad , \qquad \begin{array}{l} e = 1 \ldots v \leq (n+1)f \\ j = 0,1,\ldots s \\ q = 1 \ldots f \end{array} \tag{2.5}$$

where q refers to an end point or a corner point. In particular, $q = 1$ and $q = f$ are end points and $q = 2,3,\ldots f - 1$ are corner points. The range on the subscripts i,k,e,j, and q will be as given in equations (2.1) and (2.5) for the remainder of the material presented in this report.

The extended problem of Bolza to be considered here will be that of finding in a class of state and control variable functions

$$y_i(t) \qquad u_k(t) \qquad p_j \tag{2.6}$$

satisfying differential constraints and end point/corner conditions of the form

$$\dot{y}_i = \Phi_i(y_1 \ldots y_n, u_1 \ldots u_m, p_1 \ldots p_s, t), \tag{2.7}$$

$$\dot{p}_j = 0, \tag{2.8}$$

$$\Psi_e(t_q, y_{iq}, p_j) = 0 \quad , \tag{2.9}$$

Those which minimize a sum of the form

$$Z(t_1, t_f, y_{i1}, y_{if} p_j) + \int_{t_1}^{t_f} \left[F(y_i, u_k, p_j, t) \right] dt \quad . \tag{2.10}$$

The Necessary Optimizing Conditions

Introduction of Lagrange Multipliers - The necessary optimizing conditions for the above problem with constraints can be obtained by applying the usual method of Lagrange multipliers[3]. The assertion is made that optimal solutions to the following functional formed with the use of the multipliers μ_e, λ_i, and λ_j

$$Z^* = Z + \mu_e \Psi_e + \int_{t_1}^{t_f} \left[F + \lambda_i (\dot{y}_i - \Phi_i) + \lambda_j \dot{p}_j \right] dt, \tag{2.11}$$

subject to no constraints will also be solutions to the functional form given by equation (2.10) subject to the constraints (2.7), (2.8), and (2.9). It is noted that the functional Z^* given by equation (2.11) is not only a function of the path connecting the points q $[y_i(t), u_k(t), p_j]$ but also the quantities y_{iq}, t_q and p_j associated with the points q. The necessary optimizing conditions for extremizing Z^*, and hence the original problem, are obtained by applying the general principle that the optimizing conditions which determine the path with all of the points q fixed will remain unchanged if the points q are considered as free. Hence, two sets of optimizing criteria will, in general, have to be satisfied: conditions relating to the path and conditions relating to the points q.

Optimal path conditions - The optimizing conditions relating to the path are obtained by fixing all of the points q so that Z^* becomes

$$Z^* = \int_{t_1}^{t_f} \left[- (\lambda_i \Phi_i - F) + \lambda_i \dot{y}_i + \lambda_j \dot{p}_j \right] dt \tag{2.12}$$

If the Hamiltonian function H is defined in the usual fashion,

$$H = \lambda_i \Phi_i - F \tag{2.13}$$

Then the above integral may be written as

$$Z^* = \int_{t_1}^{t_f} \left[- H(y_i, u_k, p_j, t) + \lambda_i \dot{y}_i + \lambda_j \dot{p}_j \right] dt + C . \tag{2.14}$$

The optimal path connecting the two points t_1 and t_f may be discontinuous at the points q. Between any two points q and q + 1 the necessary conditions to extremize Z^* as given by the Euler equations from the calculus of variations are as follows[4]

$$\frac{\partial H}{\partial y_i} + \dot{\lambda}_i = 0, \qquad (2.15)$$

$$\frac{\partial H}{\partial P_j} + \dot{\lambda}_j = 0, \qquad (2.16)$$

$$\frac{\partial H}{\partial u_k} = 0. \qquad (2.17)$$

The total solution between t_1 and t_f is obtained by joining together the several continuous arcs between the points q, each of which satisfies the above Euler equations[5].

By multiplying equation (2.15) by dy_i, equation (2.16) by dp_j, and equation (2.17) by du_k and adding, it is easy to show that the above set of equations has the following first integral

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \qquad (2.18)$$

A further necessary condition for extremizing the integral given by equation (2.12) is given by certain requirements on the Weierstrass E function[6]. By reformulating this condition in terms of the notation used here and following the methods of reference 6, it is easy to show that these requirements reduce to

$$H_o > H_{NO} \qquad (2.19)$$

where H_o represents the function H evaluated with respect to optimal control and H_{NO} represents the Function H evaluated with respect to non-optimal control.

It is concluded from equation (2.19) that H takes on a maximum with respect to the control variables u_k.

<u>Optimal end point-corner conditions</u> - The optimizing conditions relating to the points q are easily obtained by noting that once a path is specified for equation (2.11), the function Z^* is then only a function of quantities y_{iq}, t_q, p_j at the points q. Hence, from the theory of ordinary maxima and minima, the necessary condition that Z^* be an extremum with respect to these quantities is given by $dZ^* = 0$.

In order to obtain an explicit expression for this condition, equation (2.11) is first written as

$$Z^* = Z + \mu_e \Psi_e = \int_1^f [-Hdt + \lambda_i dy_i + \lambda_j dp_j]. \tag{2.20}$$

Since the integrand in equation (2.20) does not necessarily represent a continuous function at the points $q = 2,3...f-1$, the interval of integration may be split into a number of subintervals with the original integral represented as a sum of integrals as follows

$$Z^* = Z + \mu_e \Psi_e + \sum_{q=1}^{f-1} \int_{q+}^{(q+1)-} [-Hdt + \lambda_i dy_i + \lambda_j dp_j] \tag{2.21}$$

where the minus sign is used to indicate that quantities are to be evaluated just previous to a corner point and the plus sign is used to indicate that quantities are to be evaluated just after a corner point. This notation is not needed for the points $q = 1$ and f .

The end point-corner conditions may now be obtained from equation (2.21) by setting $dZ^* = 0$. Carrying out this operation gives

$$\begin{aligned} dZ^* &= \frac{\partial Z}{\partial y_{i1}} dy_{i1} + \frac{\partial Z}{\partial y_{if}} dy_{if} + \frac{\partial Z}{\partial t_1} dt_1 + \frac{\partial Z}{\partial t_f} dt_f + \frac{\partial Z}{\partial p_j} dp_j \\ &+ \mu_e \frac{\partial \Psi_e}{\partial y_{iq}} dy_{iq} + \mu_e \frac{\partial \Psi_e}{\partial t_q} dt_q + \mu_e \frac{\partial \Psi_e}{\partial p_j} dp_j \\ &+ \sum_{q=1}^{f-1} \Big[- Hdt + \lambda_i dy_i + \lambda_j dp_j \Big]_{q+}^{(q+1)-} = 0 \end{aligned} \tag{2.22}$$

In abbreviated notation, equation (2.22) may be written more simply as

$$dZ^* = dZ + \mu_e d\Psi_e + \sum_{q=1}^{f-1} \Big[- Hdt + \lambda_i dy_i + \lambda_j dp_j \Big]_{q+}^{(q+1)-} = 0 \tag{2.23}$$

The formulation of the problem of Bolza has been extended in this section to include parameters defined by equation (2.8) and end point-corner constraints of the form given by equation (2.9). The corresponding Euler equations, Weierstrass condition, and end point-corner conditions as developed in this section are given by equations (2.15) - (2.19) and equation (2.23). The methods for handling restricted arcs developed in the following section will require use of the extensions presented here.

SECTION III

RESTRICTED TRAJECTORIES

Elimination of Restricted Arc

<u>Problems with restricted segments</u> - It will be assumed in the following analysis that the quantity to be extremized as a result of following an optimal trajectory can be expressed in the form

$$Z(y_{i1}, y_{if}, t_1, t_f) + \int_{t_1}^{t_f} F(y_i, u, t)dt \tag{3.1}$$

where y_i represents n state variables, u a single control variable, and t the time. It will also be assumed that the optimal trajectory will in general be composed of both restricted and unrestricted segments. Along the unrestricted segments, the dynamical and kinematical equations of constraint will be assumed to be of the form,

$$\dot{y}_i = \Phi_i(y_1, \ldots y_n, u, t), \tag{3.2}$$

where the control is completely free to be determined optimally.

A restricted segment of the trajectory is classified as one along which no optimal control law can be determined. Rather, the control is completely predetermined as a result of the statement of the restriction. The following are examples of problems which will generally yield optimal solutions with restricted segments.

1. Between two points in state space, determine the optimal trajectory which will extremize expression (3.1) subject to constraints (3.2) and which does not violate a region R in state variable space.

2. Between two points in state space, determine the optimal trajectory which will extremize expression (3.1) subject to constraints (3.2) for which a portion of the trajectory must be flown for a given length of time with the specified control u(t).

3. Between two points in state space, determine the optimal trajectory which will extremize expression (3.1) for which a portion of the trajectory is flown subject to constraints (3.2) and a portion of the trajectory must be flown according to modified constraints.

$$\dot{y}_i = \bar{\Phi}_i(y_1, \ldots y_n, t). \tag{3.3}$$

Some physical examples of problems of the type mentioned above for aircraft and rockets are given in Section IV.

For the purpose of discussion, it will be assumed that the optimal trajectory in state space for each of the above situations can be represented graphically as a curve in two dimensional space with y_i and t as the ordinate and abscissa as shown below.

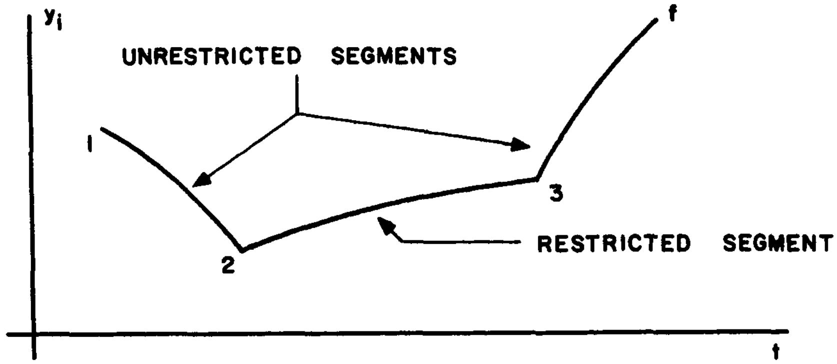

FIGURE 3.1 OPTIMAL TRAJECTORY IN STATE SPACE

For the case shown in Figure 1 the first and third segments represent unrestricted portions of the trajectory and the middle segment represents the restricted portion of the trajectory. If the segments of an optimal trajectory do not distribute themselves in this fashion or if there are more restricted and unrestricted segments than shown, this will not significantly alter the following procedure. Hence, for the sake of brevity all trajectories will be assumed to be as shown in Figure 1.

Analysis-For the trajectory shown in Figure 3.1, the functional form given by expression(3.1) may be written as follows

$$Z(y_{i1},y_{if},t_1,t_f) + \int_{1}^{2-} F(y_i,u,t)dt + \int_{2+}^{3-} F(y_i,u,t)dt + \int_{3+}^{f} F(y_i,u,t)dt. \quad (3.4)$$

Now since the control law is known along the restricted portion of the trajectory, the dynamical constraints equations (3.2) or (3.3),whichever is applicable along this portion of the trajectory,when integrated may be written as

$$\alpha_i(y_{i2},\dots y_{n2},y_{i3},\dots y_{n3},t_2,t_3) = 0 , \quad (3.5)$$

and the second integral in the expression (3.4) may be written as (with the aid of equations (3.2) or (3.3))

$$\int_{2+}^{3-} F(y_i,u,t)dt = \beta(y_{i2},y_{i3},t_2,t_3) . \quad (3.6)$$

Hence the original statement of the problem contained in equations (3.1), (3.2), and possibly (3.3) may be reformulated as follows: Extremize the functional form given by

$$Z(y_{i1},y_{if},t_1,t_f) + \beta(y_{i2},y_{i3},t_2,t_3) + \int_1^{2-} F(y_i,u,t)dt + \int_{3+}^{f} F(y_i,u,t)dt \ , \quad (3.7)$$

subject to the following dynamical and kinematical equations of constraint

$$\dot{y}_i = \Phi_i(y_1,\ldots y_n,u,t) \ , \quad (3.8)$$

plus the following "jump" conditions

$$\alpha_i(y_{i2},\ldots y_{n2},y_{i3},\ldots y_{n3},t_2,t_3) = 0 \ . \quad (3.9)$$

For the purpose of analysis, the trajectory illustrated in Figure 3.1 is now thought of as being discontinuous between the points 2 and 3. The resulting trajectory is depicted in Figure 3.2.

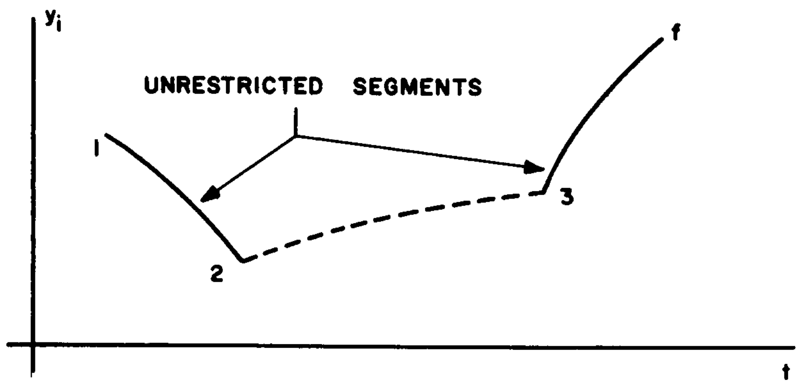

FIGURE 3.2 DISCONTINUOUS OPTIMAL TRAJECTORY WITHOUT RESTRICTED ARC

In addition to the dynamical and kinematical equations of constraint given by equation (3.2) the original problem may have had imposed, additional constraints relating to end points and/or corners. These additional constraints remain unchanged in transforming from the original problem to the reformulated one. These constraints for this problem may be written in the form,

$$\Psi_e(t_q,y_{iq}) = 0 \ . \qquad q = 1,2,3,f \quad (3.10)$$

Variations of the point 2 in general cannot be made arbitrarily (even though this is a point of discontinuity) but must be made consistent with the arc or boundary which passes through the points 2 and 3. From the statement of the problem, the equation of this boundary can usually be put in the form

$$\Theta(y_i,t) = 0 \ . \quad (3.11)$$

In particular this equation can be written as

$$\Theta(y_{i2}, t_2) = 0 \ . \tag{3.12}$$

For most situations, introduction of equation (3.12) will insure that variations of point 2 will be made consistent with the constraint $\Theta(y_i, t) = 0$ at the point 2. [An example problem involving a second order constraint given in Section IV will require a slight extension of the condition given by equation (3.12)] The jump conditions (3.9) insure that variations of the point 3 will be made consistent with $\Theta(y_i, t) = 0$ at the point 3. Equations (3.9), (3.10), and (3.12) together restrict variations of the points 1,2,3, and f.

The necessary optimizing conditions for this problem may now be obtained by using the method of Lagrange multipliers to incorporate equations (3.8), (3.9), (3.10) and (3.12) into the expression (3.7). If the usual definition of the Hamiltonian function [equation (2.13)] is used, then the following functional form is obtained.

$$Z^* = Z + \beta + \mu_e \psi_e + \nu_i \alpha_i + \pi\theta + \int_1^{2-} [-H + \lambda_i \dot{y}_i] dt + \int_{3+}^{f} [-H + \lambda_i \dot{y}_i] dt \ . \tag{3.13}$$

where $\lambda_i, \mu_e, \nu_i,$ and π are Lagrange multipliers.

Using the same arguments as in Section II, it is concluded that the following Euler equations represent necessary conditions to be satisfied by each optimal segment of the unrestricted trajectory shown in Figure 3.2.

State variable Euler Equation:

$$\frac{\partial H}{\partial y_i} + \frac{d\lambda_i}{dt} = 0 \ . \tag{3.14}$$

Control variable Euler Equation:

$$\frac{\partial H}{\partial u} = 0 \ . \tag{3.15}$$

First Integral to Euler Equations:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \ . \tag{3.16}$$

As before, the Weierstrass condition gives the further necessary condition that to minimize Z^*,H takes on a maximum with respect to the control variable u along each of the unrestricted segments.

As shown in Section II, the optimizing condition relating to the points 1, 2, 3, and f are obtained by requiring that $dZ^* = 0$. This additional requirement, using the shorthand notation of Section II may be written as

$$dZ + d\beta + \mu_e d\psi_e + \nu_i d\alpha_i + \pi d\theta + [-Hdt + \lambda_i dy_i]_1^{2-} + [-Hdt + \lambda_i dy_i]_{3+}^{f} = 0. \tag{3.17}$$

In addition to obtaining the usual transversality conditions for the end points, this latter condition is used to determine the changes in H and λ_i in "jumping" from the point 2 to the point 3. Sufficient information is contained in equations (3.14)-(3.17) to solve for a stationary trajectory which in turn is a candidate to extremize the functional (3.7). A trajectory which extremizes (3.7) will also extremize the functional (3.1) subject to the restrictions previously discussed. Several example problems are given in Section IV illustrating the use of equation (3.17).

The Use of Parameters in the Elimination of Restricted Arcs

Definition of parameters - The trajectory depicted in Figure 3.2 will not necessarily be discontinuous in all of the variables y_i and t. Whether a given variable is discontinuous or not depends upon the nature of the condition which produced the restricted arc. However all of the variables can be made continuous again at the point 2 through the introduction of parameters. If a set of parameters are defined to be equal to the value of the discontinuity for each variable between the points 2 and 3,

$$y_{i3} - y_{i2} = p_i \,, \tag{3.18}$$

$$t_3 - t_2 = p_o \,, \tag{3.19}$$

then the point 3 is in effect moved to the point 2 (Point 3 is eliminated) by replacing the variables on the final arc by

$$y_i + p_i \,, \tag{3.20}$$

$$t + p_o \,. \tag{3.21}$$

The resulting trajectory is thought of as being continuous as depicted in Figure 3.3.

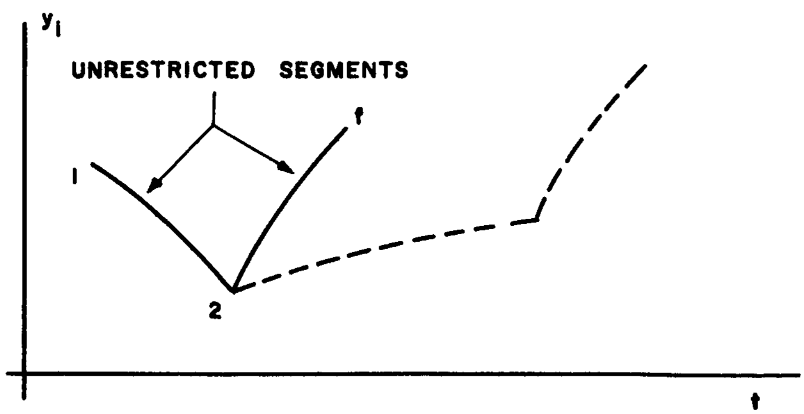

FIGURE 3.3 CONTINUOUS OPTIMAL TRAJECTORY WITHOUT RESTRICTED ARC

The original problem may now be reformulated by introducing equations (3.18)-(3.21) into the expression (3.7) to give

$$Z(y_{i1}, y_{if} + p_i, t_1, t_f + p_o) + \beta(y_{i2}, y_{i2} + p_i, t_2, t_2 + p_o)$$
$$+ \int_1^{2-} F(y_i, u, t)\,dt + \int_{2+}^{f} F(y_i + p_i, u, t + p_o)\,dt, \quad (3.22)$$

or more simply

$$Z + \beta + \int_1^{2-} F dt + \int_{2+}^{f} F dt \ . \quad (3.23)$$

In a similar manner equations (3.8), (3.9), and (3.10) may be rewritten in terms of parameters to give

$$\dot{y}_i = \phi_i(y_1, \ldots y_n, u, t) \ , \qquad t < t_2 \quad (3.24)$$

and

$$\dot{y}_i = \phi_i(y_1 + p_1, \ldots, y_n + p_n, u, t + p_o) \ , \quad t > t_2 \quad (3.25)$$

$$\alpha_i(y_{i2} + p_2, t_2 + p_o) = 0 \ , \quad (3.26)$$

$$\psi_e(t_q, y_{iq}, p_j) = 0 \ . \quad (3.27)$$

Equation (3.12) remains unchanged. If a Hamiltonian function is defined for each arc as follows

$$H^{12}(y_i, u, t) = \lambda_i \phi_i(y_1, \ldots y_n, u, t) \ - F(y_1, u, t) \ , \quad (3.28)$$

$$H^{2f}(y_i, u, t, p_j) = \lambda_i \phi_i(y_1 + p_1, \ldots y_n + p_n, u, t + p_o) - F(y_i + p_i, u, t + p_o). \quad (3.29)$$

Then the following functional form,

$$Z^* = Z + \beta + \mu_e \psi_e + \nu_i \alpha_i + \pi\theta$$

$$+ \int_1^{2-} [-H^{12} + \lambda_i \dot{y}_i + \lambda_j \dot{p}_j]dt + \int_{2+}^{f} [-H^{2f} + \lambda_i \dot{y}_i + \lambda_j \dot{p}_j]dt \quad (3.30)$$

may be used, as before, to obtain the necessary optimizing conditions for the path and corner points.

The necessary optimizing conditions - The Euler equations applicable to the arc 1-2 are given by

State Variable Euler Equation: $$\frac{\partial H^{12}}{\partial y_i} + \dot{\lambda}_i = 0 \tag{3.31}$$

Parameter Euler Equation: $$\dot{\lambda}_j = 0 \tag{3.32}$$

Control Variable Euler Equation: $$\frac{\partial H^{12}}{\partial u} = 0 \tag{3.33}$$

First Integral to Euler Equation: $$\frac{dH^{12}}{dt} = \frac{\partial H^{12}}{\partial t} \tag{3.34}$$

The Euler equations applicable to the arc 2-f are given by

State Variable Euler Equation: $$\frac{\partial H^{2f}}{\partial y_i} + \dot{\lambda}_i = 0 \tag{3.35}$$

Parameter Euler Equation: $$\frac{\partial H^{2f}}{\partial p_j} + \dot{\lambda}_j = 0 \tag{3.36}$$

Control Variable Equation: $$\frac{\partial H^{2f}}{\partial u} = 0 \tag{3.37}$$

First Integral to Euler Equation: $$\frac{dH^{2f}}{dt} = \frac{\partial H^{2f}}{\partial t} \tag{3.38}$$

In addition, in order to minimize Z* the Weierstrass condition gives that H^{12} takes on a maximum with respect to the control variable along arc 1-2 and H^{2f} takes on a maximum with respect to the control variable along the arc 2-f.

The optimizing conditions relating to the points 1,2,f are obtained using the methods of Section II by requiring that dZ* = 0. This additional requirement, using the shorthand notation of Section II may be written as

$$dZ + d\beta + \mu_e d\psi_e + \nu_i d\alpha_i + \pi d\theta$$

$$+ \Big[-H^{12}dt + \lambda_i dy_i + \lambda_j dp_j\Big]_1^{2-} + \Big[-H^{2f}dt + \lambda_i dy_i + \lambda_j dp_j\Big]_{2+}^{f} = 0 \tag{3.39}$$

An example problem illustrating the use of parameters and equations (3.39) is presented in the following section.

SECTION IV

APPLICATIONS

Brachistochronic Examples in Aircraft Flight Mechanics

Equations of motion - The determination of brachistochronic or minimum time trajectories for an aircraft present an interesting and useful class of problems which are solvable using variational methods. The methods developed in the previous section will now be applied to solve some examples which are constrained by state variable boundaries.

Following the methods of Reference 6 the dynamical and kinematical equations of motion needed for optimal trajectory analysis for an aircraft of constant mass, without induced drag and confined to flight in a vertical plane over a flat earth may be put into the following nondimensional form.

$$\frac{du}{d\tau} = r(u,\eta) - \sin\gamma, \tag{4.1}$$

$$\frac{d\xi}{d\tau} = u\cos\gamma, \tag{4.2}$$

$$\frac{d\eta}{d\tau} = u\sin\gamma, \tag{4.3}$$

where $u = \frac{v}{v_r}$ = non-dimensional velocity

$\xi = \frac{gx}{v_r^2}$ = non-dimensional range

$\eta = \frac{gx}{v_r^2}$ = non-dimensional altitude

$\tau = \frac{gt}{v_r}$ = non-dimensional time

$r = r(u,\eta) = \frac{T-D}{W}$ = non-dimensional thrust minus drag

$v_r = \frac{2W}{\rho_o A}$ = reference velocity

g = acceleration of gravity at sea level

x = range

y = altitude

t = time

T = $T(y,v)$ = specified thrust of the aircraft

D = $D(y,v)$ = drag of the aircraft

W = weight of the aircraft (constant)

ρ_o = sea level or reference density

A = wing area of aircraft.

For the aircraft described by equations (4.1) - (4.3), the variables u, ξ, and η represent state variables and γ a single control variable. The brachistochronic problem to be considered here is that of finding the minimum time trajectory from a given initial state defined by,

$$u(\tau_1) = u_1 \ , \tag{4.4}$$

$$\xi(\tau_1) = 0 \ , \tag{4.5}$$

$$\eta(\tau_1) = 0 \ , \tag{4.6}$$

to a given final state defined by

$$u(\tau_f) = \text{free} \ , \tag{4.7}$$

$$\xi(\tau_f) = \xi_f \ , \tag{4.8}$$

$$\eta(\tau_f) = \eta_f \ , \tag{4.9}$$

subject to certain other conditions which will result in restricted trajectories to be discussed in what follows.

By setting this problem up as a problem of Mayer ($F \equiv 0$, $Z = t_f - t_1$), the H function defined in Section II becomes

$$H = \lambda_u(r - \sin\gamma) + \lambda_\xi u \cos\gamma + \lambda_\eta u \sin\gamma \ , \tag{4.10}$$

and the Euler equations as developed in reference 4 become (considering only the class of trajectories where $\lambda_\xi \neq 0$, see reference 7)

u Euler Equation: $$\lambda_u' = -\lambda_u \frac{\partial r}{\partial u} - \lambda_\xi \cos\gamma - \lambda_\eta \sin\gamma \ , \tag{4.11}$$

ξ Euler Equation: $$\lambda_\xi' = 0 \ , \tag{4.12}$$

η Euler Equation: $$\lambda_\eta' = \lambda_u \frac{\partial r}{\partial \eta} \ , \tag{4.13}$$

γ Euler Equation: $$\tan \gamma = \frac{\lambda_\eta u - \lambda_u}{\lambda_\xi u} \ , \quad \lambda_\xi \neq 0 \ . \tag{4.14}$$

The following is the first integral to the above Euler Equations

$$H = \text{constant.} \tag{4.15}$$

Equations (4.1)-(4.3) and (4.11)-(4.15) must be satisfied for the unrestricted portion of the total optimal trajectory for the next three example cases.

<u>The brachistochrone problem with an altitude constraint</u> - By setting $r \equiv 0$, the problem in the preceeding section reduces to the well-known Brachistochrone problem. This problem will be solved in this section with the following altitude inequality constraint

$$\eta \geq \eta_C \ . \tag{4.16}$$

It is assumed that an attempt to solve this problem by means of equations (4.1)-(4.15) will violate the constraint as shown in Figure (4.1). Hence the optimal solution will be composed of both restricted and unrestricted arcs also depicted in Figure (4.1).

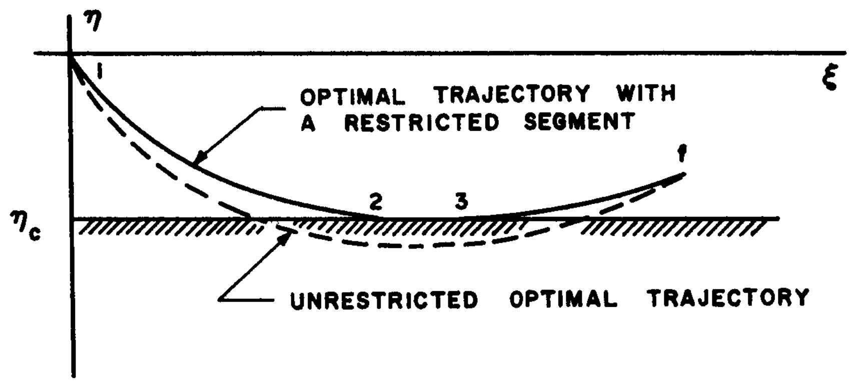

FIGURE 4.1 THE BRACHISTOCHRONE

The methods of Chapter 3 may thus be used to eliminate the arc 2-3. Following the methods of this chapter the various functional quantities given in the expression (3.13) for Z* must first be determined. Since this is a minimum time problem formulated as a problem of Bolza we have

$$Z = \tau_f \ , \tag{4.17}$$

$$\beta \equiv 0 \quad . \tag{4.18}$$

The ψ_e functions involving the endpoint conditions for this problem are obtained by rewriting equations (4.4)-(4.9) as follows,

$$\psi_1 \equiv \tau_1 = 0 \ , \tag{4.19}$$

$$\psi_2 \equiv u_1 - \text{constant} = 0 \ , \tag{4.20}$$

$$\psi_3 \equiv \xi_1 = 0 \ , \tag{4.21}$$

$$\psi_4 \equiv \eta_1 = 0 \ , \tag{4.22}$$

$$\psi_5 \equiv \xi_f - \text{constant} = 0 \ , \tag{4.23}$$

$$\psi_6 \equiv \eta_f - \text{constant} = 0 \ . \tag{4.24}$$

The α_i functions relating the "jumps" in the state variables from the point 2 to 3 are obtained from the equations of motion by observing that along the constraint the control is given by $\gamma = 0$. Thus the equations of motion can be integrated to give

$$\alpha_1 \equiv u_3 - u_2 = 0 \ , \tag{4.25}$$

$$\alpha_2 \equiv \xi_3 - \xi_2 - u_2(\tau_3 - \tau_2) = 0 \ , \tag{4.26}$$

$$\alpha_3 \equiv \eta_3 - \eta_2 = 0 \ . \tag{4.27}$$

Any alternate way of expressing these results is acceptable for the material which follows. To change the form of these expressions merely changes the algebraic manipulations to be used later on.

The function θ is obtained directly from the equation of constraint to give

$$\theta \equiv \eta_2 - \eta_C = 0 \ . \tag{4.28}$$

Thus following the procedures of Section III, dZ* may be written as follows:

$$\begin{aligned}
dZ^* = {} & d\tau_f + \mu_1 d\tau_1 + \mu_2 du_1 + \mu_3 d\xi_1 + \mu_4 d\eta_1 + \mu_5 d\xi_f + \mu_6 d\eta_f \\
& + \nu_1 \, (du_3 - du_2) + \nu_2 \, d\xi_3 - d\xi_2 - u_2 d\tau_3 + u_2 d\tau_2 - (\tau_3 - \tau_2) \, du_2 \\
& + \nu_3 \, (d\eta_3 - d\eta_2) + \pi d\eta_2 + \Big[-H d\tau + \lambda_u du + \lambda_\xi \, d\xi + \lambda_\eta d\eta \Big]_1^{2-} \\
& + \Big[-H d\tau + \lambda_u du + \lambda_\xi d\xi + \lambda_\eta d\eta \Big]_{3+}^{f} = 0 \ .
\end{aligned} \tag{4.29}$$

The method of Lagrange multipliers allows each of the deviated quantities to be considered as independent. Hence, in order to have the above entire expression equal zero each of the following equations must be satisfied.

Point 1 $d\tau$: $\mu_1 + H_1 = 0$, (4.30)

du: $\mu_2 - \lambda_{u1} = 0$, (4.31)

$d\xi$: $\mu_3 - \lambda_{\xi 1} = 0$, (4.32)

$d\eta$: $\mu_4 - \lambda_{\eta 1} = 0$. (4.33)

Point 2 $d\tau$: $\nu_2 u_2 - H_{2-} = 0$, (4.34)

du: $-\nu_1 - \nu_2(\tau_3 - \tau_2) + \lambda_{u2-} = 0$, (4.35)

$d\xi$: $-\nu_2 + \lambda_{\xi 2-} = 0$, (4.36)

$d\eta$: $-\nu_3 + \pi + \lambda_{\eta 2-} = 0$. (4.37)

Point 3 $d\tau$: $-\nu_2 u_2 + H_{3+} = 0$, (4.38)

du: $\nu_1 - \lambda_{u3+} = 0$, (4.39)

$d\xi$: $\nu_2 - \lambda_{\xi 3+} = 0$, (4.40)

$d\eta$: $\nu_3 - \lambda_{\eta 3+} = 0$. (4.41)

Point f $d\tau$: $1 - H_f = 0$, (4.42)

du: $\lambda_{uf} = 0$, (4.43)

$d\xi$: $\mu_5 + \lambda_{\xi f} = 0$, (4.44)

$d\eta$: $\mu_6 + \lambda_{\eta f} = 0$, (4.45)

Equations (4.30)-(4.33) yield no usable information, except that the initial values of $H, \lambda_u, \lambda_\xi$, and λ_η cannot be directly determined from the condition (4.29) and may have to be guessed initially.

Equation (4.34) combined with equation (4.38) yields

$$H_{3+} = H_{2-} . \quad (4.46)$$

Combining this result with equation (4.42) and equation (4.15) it is seen that

$$H = 1 \quad (4.47)$$

along each of the unrestricted portions of the trajectory.

Equation (4.35) combined with equation (4.39) yields

$$\lambda_{u3+} = \lambda_{u2-} - \nu_2 (\tau_3 - \tau_2) \ . \tag{4.48}$$

Since H = 1, the multiplier ν_2 may be evaluated from equation (4.34) to give $\nu_2 = 1/u_2$. Hence equation (4.48) may be written as:

$$\lambda_{u3+} = \lambda_{u2-} - \frac{(t_3 - t_2)}{u_2} \tag{4.49}$$

Equation (4.36) combined with equation (4.40) and the fact that $\nu_2 = 1/u_2$ yields

$$\lambda_{\xi 3+} = \lambda_{\xi 2-} = 1/u_2 \ . \tag{4.50}$$

Equation (4.37) combined with equation (4.41) yields

$$\lambda_{\eta 3+} = \lambda_{\eta 2-} + \pi \ . \tag{4.51}$$

Equations (4.44) and (4.45) yield no information directly as to the final values of λ_ξ and λ_η. Equations (4.46), (4.49) and (4.50) and (4.51) give the "jumps" in H, λ_u, λ_ξ, and λ_η in moving from point 2 to point 3.

The solution to the problem may now be obtained as follows: the condition H = 1 written out and solved for λ_u becomes

$$\lambda_u = \frac{\lambda_\eta \ u \sin\gamma + \lambda_\xi \ u \cos\gamma - 1}{\sin\gamma} \ . \tag{4.52}$$

When the above expression for λ_u is substituted into the control variable Euler equation (4.14) this equation reduces to

$$\cos\gamma = \lambda_\xi u \ . \tag{4.53}$$

Thus the optimal control depends only on one of the Lagrange multipliers λ_ξ which in this case is a constant (equation 4.12). From equation (4.50) $\lambda_\xi = 1/u_2$, hence

$$\cos\gamma_2 = 1 \ . \tag{4.54}$$

Since $u_2 = u_3$ in this case it immediately follows that

$$\cos\gamma_3 = 1 \ . \tag{4.55}$$

Thus the entrance and exit conditions to the restricted arc for this problem reduce to "tangency" conditions. That is, the unrestricted trajectory joins the restricted trajectory in a smooth fashion.

A numerical solution to the problem can now be obtained. [Note: with $r = 0$, the equations (4.1) and (4.3) may be combined and integrated using the initial conditions (4.4) and (4.6) to yield the familiar result $\frac{1}{2}u^2 + \eta = \frac{1}{2}u_1^2$. Along the line $\eta = \eta_c$, u is known and consequently λ_ξ can be determined. Hence the optimal solution can be obtained analytically. However, for the purpose of illustrating a general numerical method, it will be assumed that this result is not available.] A guess is first made for λ_ξ. The control as determined by equation (4.53) is then substituted into equations (4.1)-(4.3) [with $r = 0$] and these equations are integrated until the constraint $\eta = \eta_c$ is met. If $\lambda_\xi = 1/u_2$ at this point, then the initial trajectory computed is optimal. If not, then a new choice must be made for λ_ξ until the condition $\lambda_\xi = 1/u_2$ is satisfied at the point 2. When this condition is satisfied a guess can then be made for t_3 which determines the point 3. A "jump" to point 3 may now be made by calculating u_3, ξ_3, and η_3 from equation (4.25)-(4.27) and $\lambda_{\xi 3+}$ from equation (4.50). Evaluation of these quantities allows continuation of the integration process and if the final fixed endpoint is intercepted with the resultant trajectory, the proper choice for t_3 was made and the problem is solved. If not, t_3 must be adjusted until the end point is intercepted with the final trajectory. Variations of this procedure can be made by using the tangency condition instead of the condition $\lambda_\xi = 1/u_2$.

<u>The use of parameters in the previous application</u> - The restricted brachistochrone problem of the previous section may also be solved by using parameters to eliminate the restricted sub arc. According to equations (4.25) and (4.27) the variables u and η are continuous from point 2 to point 3. The variables ξ and τ which have a discontinuity between points 2 and 3 may be made continuous by introducing the following parameters; let

$$\tau_3 - \tau_2 = p_0 , \tag{4.56}$$

$$\xi_3 - \xi_2 = p_2 . \tag{4.57}$$

With the introduction of the above parameters the point 3 is effectively eliminated and the point 2 may be thought of as a cusp or "reflected" extremal from the line $\eta = \eta_c$ as shown below.

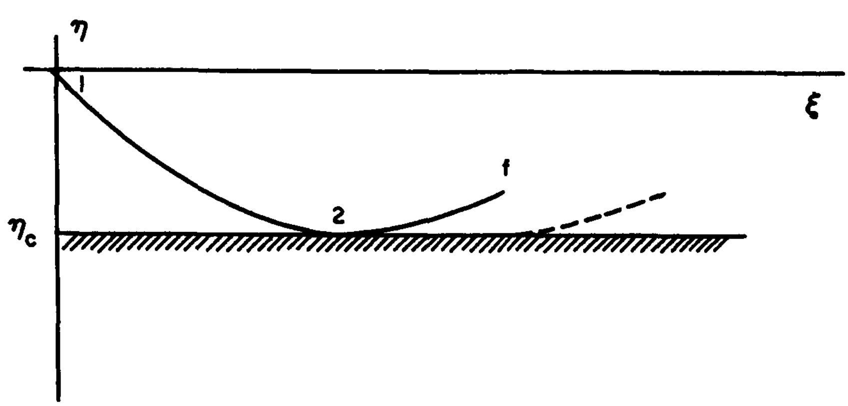

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FIGURE 4.2 CONTINUOUS BRACHISTOCHRONE WITHOUT RESTRICTED ARC

Following the methods of Section III, for this case

$$Z = \tau_f + p_0 , \tag{4.58}$$

and

$$\beta = 0 . \tag{4.59}$$

The ψ_e functions involving the endpoint conditions are written as

$$\psi_1 = \tau_1 = 0 , \tag{4.60}$$

$$\psi_2 = u_1 - C = 0 , \tag{4.61}$$

$$\psi_3 = \xi_1 = 0 , \tag{4.62}$$

$$\psi_4 = \eta_1 = 0 , \tag{4.63}$$

$$\psi_5 = \xi_f + p_2 - C = 0 , \tag{4.64}$$

$$\psi_6 = \eta_f - C = 0 . \tag{4.65}$$

The parameters p_0 and p_2 for convenience in what follows are assumed to be evaluated at point 1 although the subscripts have been omitted.

The α_i functions which previously were related to the "jump" in the state variables between the points 2 and 3 are now, with the use of parameters, relations in terms of the state variables at the point 2 and the parameters.

$$\alpha_1 \equiv 0 . \tag{4.66}$$

$$\alpha_2 \equiv p_2 - u_2 p_0 = 0 . \tag{4.67}$$

$$\alpha_3 \equiv 0 . \tag{4.68}$$

The function θ remains unchanged and is given as

$$\theta = \eta_2 - \eta_C = 0 . \tag{4.69}$$

Since the dynamical equations of constraint do not explicitly contain either ξ or τ the Hamiltonian functions for each of the arcs 1-2 and 2-f are identical and are equal to the one defined in the previous section, namely

$$H = -\lambda_u \sin \gamma + \lambda_\xi u \cos \gamma + \lambda_\eta u \sin \gamma . \tag{4.70}$$

Thus, the functional form for dZ* may be written as follows

$$dZ^* = d\tau_f + dp_0 + \mu_1 dt_1 + \mu_2 du_1 + \mu_3 d\xi_1 + \mu_4 d\eta_1 + \mu_5(d\xi_f + dp_2) + \mu_6 d\eta_f$$

$$+ \nu_2(dp_2 - u_2 dp_0 - p_0 du_2) + \pi d\eta_2 + \Big[-Hd\tau + \lambda_u du + \lambda_\xi d\xi + \lambda_\eta d\eta + \lambda_{p_0} dp_0$$

$$+ \lambda_{p_2} dp_2\Big]_1^{2-} + \Big[-Hdt + \lambda_u du + \lambda_\xi d\xi + \lambda_\eta d\eta + \lambda_{p_0} dp_0 + \lambda_{p2} dp_2\Big]_{2+}^{f} = 0 \quad (4.71)$$

For the above expression to be equal to zero each of the following equations must be satisfied:

Point 1	$d\tau$:	$\mu_1 + H_1 = 0$,	(4.72)
	du:	$\mu_2 - \lambda_{u1} = 0$,	(4.73)
	$d\xi$:	$\mu_3 - \lambda_{\xi 1} = 0$,	(4.74)
	$d\eta$:	$\mu_4 - \lambda_{\eta 1} = 0$.	(4.75)
	dp_0:	$1 - \nu_2 u_2 - \lambda_{p_{01}} = 0$,	(4.76)
	dp_2:	$\mu_5 + \nu_2 - \lambda_{p_{21}} = 0$,	(4.77)
Point 2	$d\tau$:	$-H_{2-} + H_{2+} = 0$,	(4.78)
	du:	$-\nu_2 p_0 + \lambda_{u2-} - \lambda_{u2+} = 0$,	(4.79)
	$d\xi$:	$\lambda_{\xi 2-} - \lambda_{\xi 2+} = 0$,	(4.80)
	$d\eta$:	$\pi + \lambda_{\eta 2-} - \lambda_{\eta 2+} = 0$,	(4.81)
	dp_0:	$\lambda_{p_{02-}} - \lambda_{p_{02+}} = 0$,	(4.82)
	dp_2:	$\lambda_{p_{22-}} - \lambda_{p_{22+}} = 0$,	(4.83)
Point f	$d\tau$:	$1 - H_f = 0$,	(4.84)
	du:	$\lambda_{uf} = 0$,	(4.85)

$$d\xi: \quad \mu_5 + \lambda_{\xi f} = 0 \ , \tag{4.86}$$

$$d\eta: \quad \mu_6 + \lambda_{\eta f} = 0 \ , \tag{4.87}$$

$$dp_0: \lambda_{p_{0f}} = 0 \ , \tag{4.88}$$

$$dp_2: \lambda_{p_{2f}} = 0. \tag{4.89}$$

In addition to the state variable Euler equations (4.11) - (4.14) which are valid on each unrestricted trajectory, the following parameter Euler equations are applicable (valid for each unrestricted trajectory).

$$\frac{d\lambda_{p_0}}{d\tau} = 0 \ . \tag{4.90}$$

$$\frac{d\lambda_{p_2}}{d\tau} = 0 \ . \tag{4.91}$$

Since λ_{p_0} and λ_{p_2} are constant on each subarc, continuous at point 2 and zero at the final point, then they are both zero throughout and equations (4.76) and (4.77) reduce to

$$\nu_2 = \frac{1}{u_2} \ , \tag{4.92}$$

$$\mu_5 = - \nu_2 \ . \tag{4.93}$$

It is now easy to show that the results obtained using parameters are identical with those of the previous section. As they stand, equations (4.72) - (4.75) are identical with equations (4.30) - (4.33). Equation (4.78) yields directly the equivalent combined result of the previous section given by equation (4.46). If this result is combined with equation (4.84), the result is once again obtained that

$$H = 1 \ , \tag{4.94}$$

for each portion of the unrestricted trajectory. By combining equation (4.92) with equation (4.79) the expression

$$\lambda_{u2+} = \lambda_{u2-} - \frac{p_0}{u_2} \tag{4.95}$$

is obtained which is equivalent to the combined expression (4.49) of the previous section. Combining the fact that $\frac{d\lambda_\xi}{d\tau} = 0$ with equations (4.86) (4.80) and (4.93) yields

$$\lambda_{\xi_{2-}} = \lambda_{\xi_{2+}} = \frac{1}{u_2} \tag{4.96}$$

which is equivalent to the combined result (4.50) of the previous section. Equation (4.81) yields directly the equivalent combined result (4.51) and finally equations (4.85) - (4.87) are identical with their counterparts equations (4.43) - (4.45).

Since they yield equivalent results, the methods used to solve this problem are nearly identical. As before, using a numerical integration process, the point 2 is obtained by satisfying the condition $\lambda_\xi = 1/u_2$. At this point a guess is now made for p_0 (instead of τ_3 previously) and p_2 is computed by means of equation (4.67). Integration of the constraint equations and optimizing condition may now be continued by using the condition $\xi_{2+} = \xi_{2-} + p_2$ with the ξ differential equation.

Aircraft flight mechanics with a velocity constraint. The problem to be considered in this section is the more general case for which $r = K - C_D u^2$. In addition, the aircraft is such that it must satisfy the restriction.

$$u \geq u_C \ . \tag{4.97}$$

It is assumed that an attempt to solve this problem by means of equations (4.11) - (4.15) will violate this constraint so that the optimal solution will be composed of both restricted and unrestricted arcs as shown below.

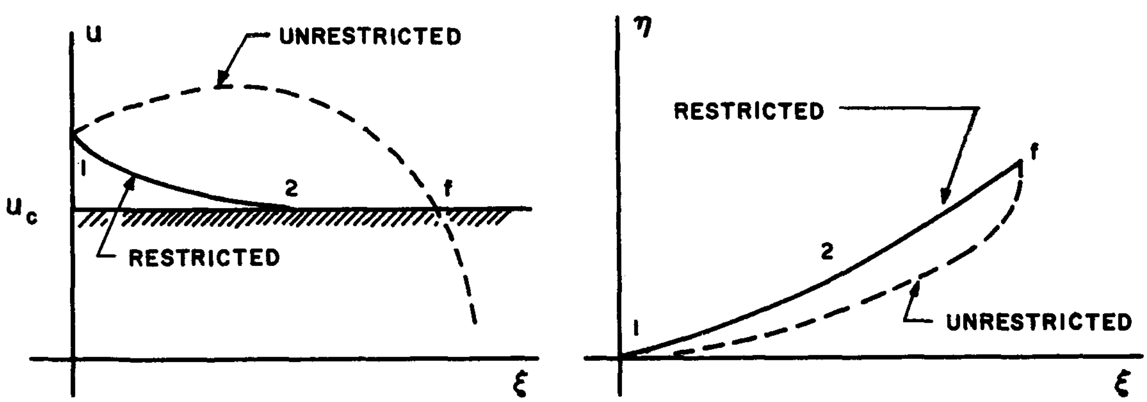

FIGURE 4.3 MINIMUM TIME RESTRICTED AIRCRAFT TRAJECTORY

The methods of Section III can be used to eliminate the arc 2-f. In this case

$$Z = \tau_f \ , \tag{4.98}$$

and

$$\beta = 0 \ , \tag{4.99}$$

The ψ_e functions involving the endpoints for this problem are written as follows:

$$\psi_1 \equiv \tau_1 = 0 \ , \tag{4.100}$$

$$\psi_2 \equiv u_1 - C = 0 \ , \tag{4.101}$$

$$\psi_3 \equiv \xi_1 = 0 \ , \tag{4.102}$$

$$\psi_4 \equiv \eta_1 = 0 \ , \tag{4.103}$$

$$\psi_5 \equiv \xi_f - C = 0 \ , \tag{4.104}$$

$$\psi_6 \equiv \eta_f - C = 0 \ . \tag{4.105}$$

The α_i functions relating the "jumps" in the state variables from the point 2 to f are obtained from the equations of motion by observing that u' = 0 along the constraint so that equation (4.1) can be solved directly for sin γ_C to give

$$\sin \gamma_C = K - C_D u_C^2 \ . \tag{4.106}$$

Equations (4.2) and (4.3) can now be solved directly for the jumps in ξ and η. The α_i functions are written as follows:

$$\alpha_1 = u_f - u_2 = 0 \ , \tag{4.107}$$

$$\alpha_2 = \xi_f - \xi_2 - u_C \cos \gamma_C \ (\tau_f - \tau_2) = 0 \ , \tag{4.108}$$

$$\alpha_3 = \eta_f - \eta_2 - u_C \sin \gamma_C \ (\tau_f - \tau_2) = 0 \ . \tag{4.109}$$

The function θ is obtained directly from the equation of constraint to give

$$\theta \equiv u_2 - u_C = 0 \ . \tag{4.110}$$

Thus the function dZ* may be written as follows

$$\begin{aligned} dZ^* = {} & d\tau_f + \mu_1 d\tau_1 + \mu_2 du_1 + \mu_3 d\xi_1 + \mu_4 d\eta_1 + \mu_5 d\xi_f + \mu_6 d\eta_f \\ & + \nu_1(du_f - du_2) + \nu_2[d\xi_f - d\xi_2 - u_C \cos \gamma_C \ (d\tau_f - d\tau_2)] \\ & + \nu_3 \ [d\eta_f - d\eta_2 - u_C \sin \gamma_C \ (d\tau_f - d\tau_2)] \ + \pi du_2 \\ & + \Big[-Hd\tau + \lambda_u du + \lambda_\xi d\xi + \lambda_\eta d\xi\Big]_1^{2-} = 0 \ . \end{aligned} \tag{4.111}$$

Considering each of the different differentials as being independent yields the following results:

Point 1

$$d\tau: \quad \mu_1 + H_1 = 0 \, , \tag{4.112}$$

$$du: \quad \mu_2 - \lambda_{u1} = 0 \, , \tag{4.113}$$

$$d\xi: \quad \mu_3 - \lambda_{\xi 1} = 0 \, , \tag{4.114}$$

$$d\eta: \quad \mu_4 - \lambda_{\eta 1} = 0 \, , \tag{4.115}$$

Point 2

$$d\tau: \quad \nu_2 u_C \cos\gamma_C + \nu_3 u_C \sin\gamma_C - H_{2-} = 0 \, , \tag{4.116}$$

$$du: \quad -\nu_1 + \pi + \lambda_{u2-} = 0 \, , \tag{4.117}$$

$$d\xi: \quad -\nu_2 + \lambda_{\xi 2-} = 0 \, , \tag{4.118}$$

$$d\eta: \quad -\nu_3 + \lambda_{\eta 2-} = 0 \, , \tag{4.119}$$

Point f

$$d\tau: \quad 1 - \nu_2 u_C \cos\gamma_C - \nu_3 u_C \sin\gamma_C = 0 \, , \tag{4.120}$$

$$du: \quad \nu_1 = 0 \, , \tag{4.121}$$

$$d\xi: \quad \mu_5 + \nu_2 = 0 \, , \tag{4.122}$$

$$d\eta: \quad \mu_6 + \nu_3 = 0 \, . \tag{4.123}$$

As in the previous example, equations (4.112) - (4.115) yield no usable information. Combining equations (4.116) with (4.120) yields

$$H_2 = 1 \, . \tag{4.124}$$

Combining this result with equation (4.19) it is seen that

$$H = 1 \tag{4.125}$$

along the unrestricted portion of the trajectory. Combining the result of equation (4.121) with equation (4.117) yields

$$\lambda_{u2-} = -\pi \, . \tag{4.126}$$

Equations (4.118) and (4.122) combine to give

$$\lambda_{\xi 2-} = -\mu_5 \, , \tag{4.127}$$

and equations (4.119) and (4.123) combine to give

$$\lambda_{\eta 2-} = -\mu_6 \ . \tag{4.128}$$

If equations (4.118) and (4.119) are substituted into equation (4.120) the following result is obtained

$$1 - \lambda_{\xi 2-} u_C \cos \gamma_C - \lambda_{\eta 2-} u_C \sin \gamma_C = 0 \ . \tag{4.129}$$

The solution to the problem considered in this section may now be obtained. The Euler equations (4.12) and (4.13) in this case reduce to

$$\lambda_\xi = \text{constant} = a \ , \tag{4.130}$$

$$\lambda_\eta = \text{constant} = b \ . \tag{4.131}$$

Equation (4.129) may be written as

$$a u_C \cos \gamma_C + b u_C \sin \gamma_C = 1 \ . \tag{4.132}$$

Since H = 1 along the unrestricted trajectory, the first integral expression (4.15) may be written as follows

$$\lambda_u (r - \sin \gamma) + a u \cos \gamma + b u \sin \gamma = 1 \ . \tag{4.133}$$

Now since this expression must hold throughout the unrestricted trajectory, in particular, it must hold when u first reaches the value u_C. Under such circumstance, the above equation reduces to

$$\lambda_u [K - C_D {u_C}^2 - \sin \gamma_{2-}] + a u_C \cos \gamma_{2-} + b u_C \sin \gamma_{2-} = 1 \tag{4.134}$$

This equation will satisfy the requirement given by equation (4.132) for joining the restricted segment provided that

$$\gamma_{2-} = \gamma_C \tag{4.135}$$

Thus, as with the preceding example the joining condition is a tangency condition. In this case, condition (4.135) is the most convenient one to use for determining the location of point 2. The unrestricted trajectory may be generated using the methods of reference 6. When the boundary $u = u_C$ is met, condition (4.135) is used to determine if the choices made for the initial conditions were proper. With the correct choices for the initial conditions, since the restricted segment is also the final segment, if this segment passes through the desired endpoint, then the final time τ_f can be calculated from equation (4.108) or (4.109) and the problem is solved.

An Example With A Second Order Constraint

Equations of constraint - An n th order constraint is defined as follows: For a given inequality constraint of the form

$$S(y_i t) \geq 0 \ , \tag{4.136}$$

if $\frac{d^n S}{dt^n}$ is the first derivative of S which explicitly contains the control variable (upon substituting the dynamical equations of constraint into $\frac{d^n S}{dt^n}$) Then S is called an n th order constraint.

An illustrative example problem involving a second order constraint has been worked out by Bryson[8] who applies a general method, which requires the integration of the Euler equations along the restricted arc. The following analysis shows that this same problem can also be solved by eliminating the restricted arc as discussed in Section III.

The example problem is formulated as follows: minimize E_f subject to the following differential equations of constraint,

$$\dot{E} = \frac{1}{2} a^2 \ , \tag{4.137}$$

$$\dot{v} = a \ , \tag{4.138}$$

$$\dot{x} = v \ , \tag{4.139}$$

and the inequality

$$S \equiv x - \ell \leq 0 \ . \tag{4.140}$$

The initial values of E, v, x, and t as well as the final values of x, v and t are given. In this case there are three state variables E, v, and x and a single control variable a, leaving one degree of freedom. The constraint equation (4.140) is of second order since

$$\ddot{S} = \ddot{x} = \dot{v} = a \ , \tag{4.141}$$

is the first derivative of S in which the control variable "a" appears.

If this problem is set up as a problem of Mayer ($z = E_f$, $F \equiv 0$) then the Hamiltonian function as defined in Section II becomes

$$H = \lambda_E \frac{1}{2} a^2 + \lambda_v a + \lambda_x v \ . \tag{4.142}$$

The Euler Equations with this H function are as follows:

E Euler Equation: $\dot{\lambda}_E = 0 \ ,$ (4.143) v Euler Equation: $\dot{\lambda}_v = -\lambda_x$, (4.144)

x Euler Equation: $\dot{\lambda}_x = 0$, (4.145)

a Euler Equation: $a = -\frac{\lambda_v}{\lambda_E}$. (4.146)

With the following first integral to the Euler Equations

$$H = \text{Constant} . \quad (4.147)$$

Restructed solution - It is now assumed that an attempt to solve this problem by means of the above equations with a given set of initial and final conditions will violate the inequality constraint as shown in Figure 4.4. Hence the optimal solution will be composed of both restricted and unrestricted arcs.

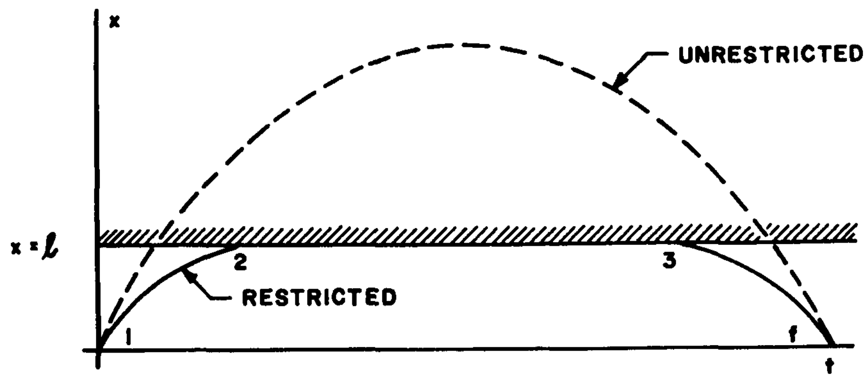

FIGURE 4.4 RESTRICTION DUE TO SECOND ORDER CONSTRAINT

Following the methods of Section III, the arc 2-3 may now be eliminated. In this case

$$z = E_f , \quad (4.148)$$

$$\beta = 0 . \quad (4.149)$$

The following initial and final conditions will be chosen for this example

$$\psi_1 \equiv t_1 = 0 , \quad (4.150)$$

$$\psi_2 \equiv H_1 = 0 , \quad (4.151)$$

$$\psi_3 \equiv v_1 - 1 = 0 , \quad (4.152)$$

$$\psi_4 \equiv x_1 = 0 , \quad (4.153)$$

$$\psi_5 \equiv t_f - 1 = 0 \,, \tag{4.154}$$

$$\psi_6 \equiv v_f + 1 = 0 \,, \tag{4.155}$$

$$\psi_7 \equiv x_f = 0 \,. \tag{4.156}$$

The "jumps" in the state variables from the point 2 to 3 are obtained from the dynamical equations of constraint by observing that along the constraint the control $a = 0$.

$$\alpha_1 \equiv E_3 - E_2 = 0 \,, \tag{4.157}$$

$$\alpha_2 \equiv v_3 - v_2 = 0 \,, \tag{4.158}$$

$$\alpha_3 \equiv x_3 - x_2 = 0 \,. \tag{4.159}$$

In this case in order to make variations of point 2 consistent with the boundary, two conditions must be satisfied.

$$\theta_1 \equiv x_2 - \ell = 0 \,, \tag{4.160}$$

$$\theta_2 \equiv v_2 = 0 \,. \tag{4.161}$$

The dZ* function becomes

$$\begin{aligned} dZ^* = {} & dE_f + \mu_1 dt_1 + \mu_2 dE_1 + \mu_3 dv_1 + \mu_4 dx_1 + \mu_5 dt_f + \mu_6 dv_f + \mu_7 dx_f \\ & + \nu_1(dE_3 - dE_2) + \nu_2(dv_3 - dv_2) + \nu_3(dx_3 - dx_2) + \pi_1 dx_2 + \pi_2 dv_2 \\ & + \Big[-Hdt + \lambda_E dE + \lambda_v dv + \lambda_x dx\Big]_1^{2-} \\ & + \Big[-Hdt + \lambda_E dE + \lambda_v dv + \lambda_x dx\Big]_{3+}^{f} = 0 \,. \end{aligned} \tag{4.162}$$

By requiring that each of the differential expressions be equal to zero results in:

Point 1 dt: $$\mu_1 + H_1 = 0 \,, \tag{4.163}$$

dE: $$\mu_2 - \lambda_{E1} = 0 \,, \tag{4.164}$$

dv: $$\mu_3 - \lambda_{v1} = 0 \,, \tag{4.165}$$

dx: $$\mu_4 - \lambda_{x1} = 0 \,, \tag{4.166}$$

Point 2 dt: $-H_{2-} = 0$, (4.167)

dE: $-\nu_1 + \lambda_{E2-} = 0$, (4.168)

dv: $-\nu_2 + \lambda_{v2-} + \pi_2 = 0$, (4.169)

dx: $-\nu_3 + \lambda_{x2-} + \pi_1 = 0$, (4.170)

Point 3 dt: $H_{3+} = 0$, (4.171)

dE: $\nu_1 - \lambda_{E3+} = 0$, (4.172)

dv: $\nu_2 - \lambda_{v3+} = 0$, (4.173)

dx: $\nu_3 - \lambda_{x3+} = 0$, (4.174)

Point 4 dt: $\mu_5 - H_f = 0$, (4.175)

dE: $1 + \lambda_{Ef} = 0$, (4.176)

dv: $\mu_6 + \lambda_{vf} = 0$, (4.177)

dx: $\mu_7 + \lambda_{xf} = 0$. (4.178)

Equations (4.163) - (4.166) yield no usable information. The initial values of H, λ_E, λ_v, and λ_x cannot be determined directly from condition (4.162).

Since H is constant along each of the unrestricted trajectories, equations (4.167) and (4.170) require that

$$H = 0 \tag{4.179}$$

along arcs 1-2 and 3-f. Equations (4.168) and (4.172) may be combined to yield

$$\lambda_{E3+} = \lambda_{E2-} \ . \tag{4.180}$$

Equations (4.169) and (4.173) may be combined to yield

$$\lambda_{v3+} = \lambda_{v2-} + \pi_2 \ . \tag{4.181}$$

Equations (4.170) and (4.174) may be combined to yield

$$\lambda_{x3+} = \lambda_{x2-} + \pi_1 \ . \tag{4.182}$$

Equations (4.175), (4.177) and (4.178) yield no usable information. Equation (4.176) gives directly the useful result $\lambda_{Ef} = -1$. Combining this result with equations (4.143) and (4.180) yields

$$\lambda_E = -1 \tag{4.183}$$

along each of the arcs 1-2 and 3-f.

The solution to the problem may now be obtained by integrating the Euler equations in conjunction with the equations of motion. According to the Euler equation (4.145) λ_x is a constant along each of the arcs 1-2 and 3-f. If b is this constant along the first arc

$$\lambda_x = b, \qquad 0 \leq t \leq t_2 \tag{4.184}$$

Then according to equation (4.182) along the second arc the constant is $b + \pi$, or alternately

$$\lambda_x = d \; . \qquad t_3 \leq t \leq 1 \tag{4.185}$$

This information may now be used to integrate the Euler equation (4.144) to give

$$\lambda_v = -bt + \lambda_{v1} \; , \qquad 0 \leq t \leq t_2 \tag{4.186}$$

$$\lambda_v = -d(t - t_3) + \lambda_{v3+} \; . \qquad t_3 \leq t \leq 1 \tag{4.187}$$

These results plus equation (4.183) may now be substituted into the Euler equation (4.146) to yield the optimal control law along each of the unrestricted sub arcs.

$$a = \lambda_{v1} - bt \quad , \qquad 0 \leq t \leq t_2 \tag{4.188}$$

$$a = \lambda_{v3+} - d(t - t_3) \; . \qquad t_3 \leq t \leq 1 \tag{4.189}$$

Substituting $\lambda_v = a$, $\lambda_E = -1$, and $\lambda_x = b$ or $\lambda_x = d$ into the H function equation (4.142) with the appropriate optimal control equation (4.188) or (4.189) yields

$$H = \frac{1}{2} (\lambda_{v1} - bt)^2 + bv \qquad 0 \leq t \leq t_2 \tag{4.190}$$

$$H = \frac{1}{2} [\lambda_{v3+} - d(t - t_3)]^2 + dv \qquad t_3 \leq t \leq 1 \tag{4.191}$$

Since H = 0 throughout the unrestricted arcs, the following results are obtained by setting H = 0 at:

$$\text{Point 1} \quad [t = 0, v_1 = 1] \qquad \lambda_{v1}^2 = -2b \tag{4.192}$$

Point 2 $\quad [t = t_2 \ , \ v_2 = 0] \quad t_2 = \frac{\lambda_{v1}}{b}$ (4.193)

[Note: From equations (4.158) and (4.161) $v_2 = v_3 = 0$]

Point 3 $\quad [t = t_3 \ , \ v_3 = 0] \quad \lambda_{v3+} = 0$ (4.194)

Point 4 $\quad [t = 1 \ , \ v_f = -1] \quad \frac{1}{2}[-d(1 - t_3)]^2 - d = 0$ (4.195)

By substituting equation (4.193) into equation (4.188) and equation (4.194) into equation (4.189) it is seen that the optimal control just previous to the constrained arc at the point 2 is zero and likewise at the point 3.

With the optimal control law given by equations (4.188) and (4.189) equations (4.138) and (4.139) may be integrated to yield:

$$v = -\frac{1}{2b}(\lambda_{v1} - bt)^2 \ , \qquad 0 \le t \le t_2 \tag{4.196}$$

$$v = -\frac{d}{2}(t - t_3)^2 \ , \qquad t_3 \le t \le 1 \tag{4.197}$$

$$x = \frac{1}{6b^2}(\lambda_{v1} - bt)^3 + \frac{\lambda_{v1}^3}{6b^2} \ , \quad 0 \le t \le t_2 \tag{4.198}$$

$$x = \ell - \frac{d}{6}(t - t_3)^3 \ . \qquad t_3 \le t \le 1 \tag{4.199}$$

Applying the condition that $x = \ell$ at $t = t_2 = \frac{\lambda_{v1}}{b}$ to equation (4.198) yields:

$$b = -\frac{2}{9\ell^2} \tag{4.200}$$

Thus from equation (4.192)

$$\lambda_{v1} = -\frac{2}{3\ell} \ . \tag{4.201}$$

and from equation (4.193)

$$t_2 = 3\ell \ . \tag{4.202}$$

Applying the condition that $x = 0$ when $t = 1$ to equation (4.199) yields

$$d = \frac{6\ell}{(1-t_3)^3} \ . \tag{4.203}$$

Thus from equation (4.195)

$$t_3 = 1 = 3\ell \quad . \tag{4.204}$$

The time spent on the restricted arc is given by

$$t_3 - t_2 = 1 - 6\ell \quad . \tag{4.205}$$

It is noted from this equation that the time spent on the arc goes to zero when $\ell = \frac{1}{6}$. It is concluded that for $0 \leq \ell < \frac{1}{6}$ a finite length of time will be spent on the restricted arc. The optimal solution in this case is given by the following sets of equations.

For $0 \leq t \leq 3\ell$:

$$a = -\frac{2}{3\ell}\left(1 - \frac{t}{3\ell}\right) \tag{4.206}$$

$$E = \frac{2}{9\ell}\left[1-\left(1 - \frac{t}{3\ell}\right)^3\right] \tag{4.207}$$

$$v = \left(1 - \frac{t}{3\ell}\right)^2 \tag{4.208}$$

$$x = \ell \left[1 - \left(1 - \frac{t}{3\ell}\right)^3\right] \tag{4.209}$$

For $3\ell \leq t \leq 1 - 3\ell$

$$a = 0 \tag{4.210}$$

$$E = \frac{2}{9\ell} \tag{4.211}$$

$$v = 0 \tag{4.212}$$

$$x = \ell \tag{4.213}$$

For $1 - 3\ell \leq t \leq 1$

$$a = -\frac{2}{3\ell}\left(1 - \frac{1-t}{3\ell}\right) \tag{4.214}$$

$$E = \frac{2}{9\ell}\left[1 + \left(1 - \frac{1-t}{3\ell}\right)^3\right] \qquad (E_{min} = 4/9\ell) \tag{4.215}$$

$$v = -\left(1 - \frac{1-t}{3\ell}\right)^2 \tag{4.216}$$

$$x = \ell\left[1 - \left(1 - \frac{1-t}{3\ell}\right)^3\right] \tag{4.217}$$

Unrestricted solution. An interesting feature of this problem is the fact that ℓ must be greater than 1/4 in order for the resultant trajectory to be unrestricted. This is easy to show by simply integrating the constraint and Euler equations without imposing the conditions given by equations (4.167) - (4.174). [Points 2 and 3 do not exist]

The endpoint condition (4.176) requires that $\lambda_E = -1$ as before, and the Euler equation (4.145) requires that λ_x be constant throughout the trajectory. The optimal control equation then becomes

$$a = -\lambda_x t + \lambda_{v1} \; . \tag{4.218}$$

The constraint equations (4.138) and (4.139) may now be integrated to yield

$$v - v_1 = \lambda_{v1} t - \frac{\lambda_x}{2} \quad , \tag{4.219}$$

$$x - x_1 = t + \frac{\lambda_{v1}}{2} t^2 - \frac{\lambda_x}{2} \; . \tag{4.220}$$

Applying the boundary conditions that at $t = 0$, $v_1 = 1$, $x_1 = 0$, and at $t = 1$, $v_f = -1$, $x_f = 0$, yields the results

$$\lambda_x = 0 \tag{4.221}$$

$$\lambda_{v1} = -2 \tag{4.222}$$

Thus the optimal solution is easily obtained in this case and is given by the following set of equations

$$a = -2 \quad , \tag{4.223}$$

$$E = 2t \quad , \tag{4.224}$$

$$v = 1 - 2t \; , \tag{4.225}$$

$$x = t(1 - t) \; . \tag{4.226}$$

It is noted from equation (4.226) that the maximum value of x is 1/4. Hence if $\ell > 1/4$ the inequality constraint will not be encountered.

Reflected solution. For $0 \leq \ell \leq 1/6$ a restricted solution is obtained, and for $\ell > 1/4$ an unrestricted solution is obtained. Hence it is concluded that between $\ell = 1/6$ and $\ell = 1/4$, the extremal curve is a reflected one. For a reflected extremal there is no point 3 since $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence the condition yielding the transversality corner conditions reduces to

$$dZ^* = dE_f + \mu_1 dt_1 + \mu_2 dE_2 + \mu_3 dv_1 + \mu_4 dx_1 + \mu_5 dt_f + \mu_6 dv_f + \mu_7 dx_f$$
$$+ \pi_1 dx_2 + \pi_2 dv_2 + \Big[-Hdt + \lambda_E dE + \lambda_v dv + \lambda_x dx \Big]_1^{2-}$$
$$+ \Big[-Hdt + \lambda_E dE + \lambda_v dv + \lambda_x dx \Big]_{2+}^{f} = 0 \ . \quad (4.227)$$

The endpoints conditions are the same as before and are given by equations (4.163) - (4.166) and (4.175) - (4.178). The conditions at point 2 become

Point 2 dt: $-H_{2-} + H_{2+} = 0 \ ,$ (4.228)

dE: $\lambda_{E2-} - \lambda_{E2+} = 0 \ ,$ (4.229)

dv: $\lambda_{v2-} - \lambda_{v2+} + \pi_2 = 0 \ ,$ (4.230)

dx: $\pi_1 + \lambda_{x2-} - \lambda_{x2+} = 0 \ .$ (4.231)

As before, it is concluded that,

$$\lambda_E = -1 \ . \quad (4.232)$$

According to equation (4.145) λ_x is a constant along each of the arcs 1-2 and 2-f. If b is this constant along the first arc

$$\lambda_x = b \qquad 0 \leq t < t_2 \quad (4.233)$$

Then according to equation (4.231) along the second arc the constant is $b + \pi_1$ or alternatly

$$\lambda_x = d \ . \qquad t_2 < t \leq 1 \quad (4.234)$$

This information may now be used to integrate the Euler equation (4.144) to give

$$\lambda_v = -bt + \lambda_{v1} \ , \qquad 0 \leq t \leq t_2 \quad (4.235)$$

$$\lambda_v = -d(t - t_2) + \lambda_{v2+} \ . \qquad t_2 < t \leq 1 \quad (4.236)$$

These results plus the result $\lambda_E = -1$ may now be substituted into the Euler equation (4.146) to yield the optimal control law along each of the unrestricted sub arcs

$$a = \lambda_{v1} - bt \quad , \qquad 0 \leq t \leq t_2 \tag{4.237}$$

$$a = \lambda_{v2+} - d(t - t_2) \ . \qquad t_2 \leq t \leq 1 \tag{4.238}$$

The H function with the proper optimal control law for each arc becomes

$$H = \frac{1}{2}(\lambda_{v1} - bt)^2 + bv \qquad 0 \leq t \leq t_2 \tag{4.239}$$

$$H = \frac{1}{2}[\lambda_{v2+} - d(t - t_2)]^2 + vd \qquad t_2 < t < 1 \tag{4.240}$$

Applying the condition (4.228), making note of equation (4.161) yields

$$\lambda_{v2+} = \lambda_{v1} - bt_2 \ . \tag{4.241}$$

If the control law for each segment 1-2 and 2-f is substituted into the equations of constraint (4.138) and (4.139) and these equations integrated, the resultant equations evaluated at point 2 yield

$$1 + \lambda_{v1}t_2 - \frac{bt_2^2}{2} = 0 \quad , \tag{4.242}$$

$$t_2(1 + \frac{1}{2}\lambda_{v1}t_2 - \frac{1}{6}bt_2^2) = \ell \quad , \tag{4.243}$$

and evaluated at point f yield

$$1 + \lambda_{v2+}(1 - t_2) - \frac{1}{2}d(1 - t_2)^2 = 0 \ , \tag{4.244}$$

$$\ell + \frac{1}{2}\lambda_{v2+}(1 - t_2)^2 - \frac{1}{6}d(1 - t_2)^3 = 0 \ . \tag{4.245}$$

The five equations (4.241) - (4.245) may now be solved for the five unknown constants, t_2, λ_{v1}, λ_{v2+}, b, and d.

$$t_2 = 1/2 \quad , \tag{4.246}$$

$$\lambda_{v1} = -8(1 - 3\ell) \quad , \tag{4.247}$$

$$\lambda_{v2+} = 4(1 - 6\ell) \quad , \tag{4.248}$$

$$b = -24(1 - 4\ell) \quad , \tag{4.249}$$

$$d = 24(1 - 4\ell) \quad . \tag{4.250}$$

with these constants evaluated the solution is easily found to be:

For $0 \leq t \leq t_2 = t_3 = \frac{1}{2}$

$$a = -8(1 - 3\ell) + 24(1 - 4\ell)t \ , \tag{4.251}$$

$$v = 1 - 8(1 - 3\ell)t + 12(1 - 4\ell)t^2 \ , \tag{4.252}$$

$$x = t - 4(1 - 3\ell)t^2 + 4(1 - 4\ell)t^3 \ , \tag{4.253}$$

For $\frac{1}{2} \leq t \leq 1$

$$a = -8(1 - 3\ell) + 24(1 - 4\ell)(1 - t) \ , \tag{4.254}$$

$$v = 1 - 8(1 - 3\ell)(1 - t) + 12(1 - 4\ell)(1 - t)^2 \ , \tag{4.255}$$

$$x = 1 - t - 4(1 - 3\ell)(1 - t)^2 + 4(1 - 4\ell)(1 - t)^3 \ , \tag{4.256}$$

$$E_{min} = 8(1 - 6\ell + 12\ell^2) \ . \tag{4.257}$$

These results agree with reference 8.

Examples in Rocket Flight Mechanics

Case I -- Optimal staging with coast periods - In a recent article by Mason[9] a variational method for determining the optimal stage sizes for multistage rockets is presented. In addition to the optimal steering program this method simultaneously yields the optimal propellent and structure weights for each stage. The following example illustrates how the methods of Section III may be applied to the problem discussed by Mason with the additional condition that a finite coast period takes place after each stage allowing time for the burned out stage to be discarded.

Consider a multistage rocket vehicle in flight above the atmosphere with constant thrust and fuel flow rate over a flat earth with uniform gravity as shown below

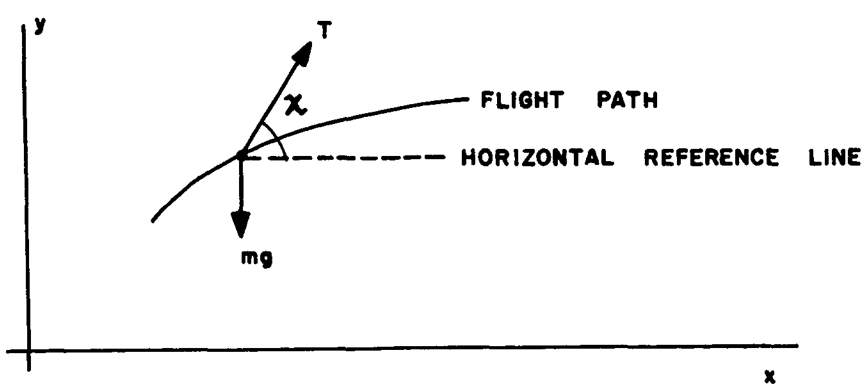

FIGURE 4.5 APPLIED FORCES ON ROCKET

The corresponding equations of motion are:

$$\dot{u} = \frac{T}{m} \cos \chi \quad , \tag{4.258}$$

$$\dot{v} = \frac{T}{m} \sin \chi - g \quad , \tag{4.259}$$

$$\dot{x} = u \quad , \tag{4.260}$$

$$\dot{y} = v \quad , \tag{4.261}$$

$$\dot{m} = -\beta \quad , \tag{4.262}$$

where g is the acceleration of gravity at the surface of the earth and T and β are given constants which may have different values for different stages. The velocity components in the x and y directions are u and v and m is the mass of the rocket. The state variables x, y, u, v, and m are subject to the single control variable χ which is the direction of the thrust with respect to the x axis. This leaves one degree of freedom for optimal control.

It is assumed that the trajectory consists of three thrusting stages separated by coasting periods of finite length. This assumption will require that the optimal solution has restricted segments which are depicted in Figure 4.6. This problem is of type 3 as discussed in the beginning of Section III.

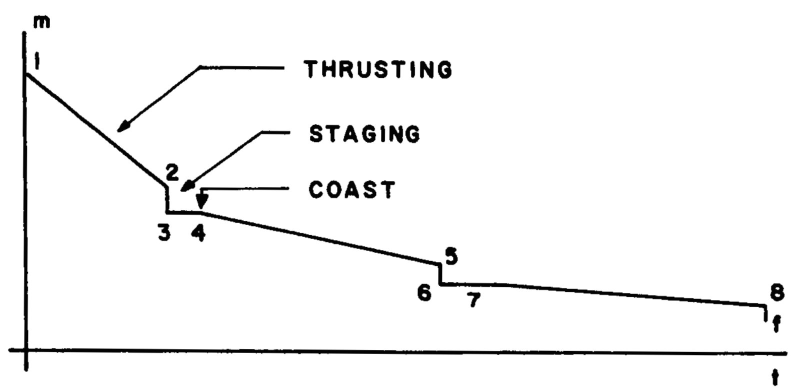

FIGURE 4.6 MASS CHANGES FOR A MULTISTAGE ROCKET

The structural weight of each stage will be assumed proportional to the fuel weight and, hence, the burning time of that stage (since the fuel flow is constant). The quantity to be maximized is the payload.

If this problem is set up as a problem of Mayer ($Z = m_f$, $F = 0$) then the Hamiltonian function as defined in Section II becomes

$$H = \lambda_u \frac{T}{m} \cos\chi + \lambda_v \left(\frac{T}{m} \sin\chi - g\right) + \lambda_x u + \lambda_y v - \lambda_m \beta \quad . \tag{4.263}$$

The Euler equations applicable for arcs 1-2, 4-5, and 7-8 are as follows

u Euler Equation $\quad \dot{\lambda}_u = -\lambda_x \quad ,$ (4.264)

v Euler Equation $\quad \dot{\lambda}_v = -\lambda_y \quad ,$ (4.265)

x Euler Equation $\quad \dot{\lambda}_x = 0 \quad ,$ (4.266)

y Euler Equation $\quad \dot{\lambda}_y = 0 \quad ,$ (4.267)

m Euler Equation $\quad \dot{\lambda}_m = \lambda_u \frac{T}{m^2} \cos\chi + \lambda_v \frac{T}{m^2} \sin\chi \quad ,$ (4.268)

χ Euler Equation $\quad \tan\chi = \frac{\lambda_v}{\lambda_u} \quad .$ (4.269)

With the following first integral to the Euler equations

$$H = \text{constant} \ .$$

Following the methods of Section III the arcs 2-4, 5-7 and 8-f may now be eliminated. In this case

$$z = m_f \quad , \tag{4.270}$$

$$\beta = 0 \ . \tag{4.271}$$

The following conditions are applicable at the various points along the trajectory.

$$\psi_1 = t_1 = 0 \ , \tag{4.272}$$

$$\psi_2 = u_1 - C = 0 \ , \tag{4.273}$$

$$\psi_3 = v_1 - C = 0 \ , \tag{4.274}$$

$$\psi_4 = x_1 = 0 \ , \tag{4.275}$$

$$\psi_5 = y_1 = 0 \ , \tag{4.276}$$

$$\psi_6 = m_1 - C = 0 \ , \tag{4.277}$$

$$\psi_7 = m_4 - m_2 + k_1 t_2 = 0 , \qquad (4.278)$$

$$\psi_8 = m_7 - m_5 + k_2(t_5 - t_4) = 0 , \qquad (4.279)$$

$$\psi_9 = m_f - m_8 + k_3(t_8 - t_7) = 0 , \qquad (4.280)$$

$$\psi_{10} = u_f - C = 0 , \qquad (4.281)$$

$$\psi_{11} = v_f - C = 0 , \qquad (4.282)$$

$$\psi_{12} = y_f - C = 0 , \qquad (4.283)$$

$$\psi_{13} = t_4 - t_2 - C = 0 , \qquad (4.284)$$

$$\psi_{14} = t_7 - t_5 - C = 0 . \qquad (4.285)$$

The "jumps" in the state variables (other than mass which has already been specified) from the points 2-4, 5-7 and 8-f may be obtained from the dynamical equations of constraint by setting T = 0 (which in effect yield a new set of dynamical equations with no control) to yield

$$\alpha_1 = u_4 - u_2 = 0 , \qquad (4.286)$$

$$\alpha_2 = v_4 - v_2 + g(t_4 - t_2) = 0 , \qquad (4.287)$$

$$\alpha_3 = x_4 - x_2 - u_2(t_4 - t_2) = 0 , \qquad (4.288)$$

$$\alpha_4 = y_4 - y_2 - v_2(t_4 - t_2) + \frac{1}{2} g(t_4 - t_2)^2 = 0 , \qquad (4.289)$$

$$\alpha_5 = u_7 - u_5 = 0 , \qquad (4.290)$$

$$\alpha_6 = v_7 - v_5 + g(t_7 - t_5) = 0 , \qquad (4.291)$$

$$\alpha_7 = x_7 - x_5 - u_5(t_7 - t_5) = 0 , \qquad (4.292)$$

$$\alpha_8 = y_7 - y_5 - v_5(t_7 - t_5) + \frac{1}{2} g(t_7 - t_5) = 0 , \qquad (4.293)$$

$$\alpha_9 = u_f - u_8 = 0 , \qquad (4.294)$$

$$\alpha_{10} = v_f - v_8 = 0 , \qquad (4.295)$$

$$\alpha_{11} = x_f - x_8 = 0 , \qquad (4.296)$$

$$\alpha_{12} = y_f - y_8 = 0 . \qquad (4.297)$$

In this case the function θ is identically zero. By forming Z^* and setting $dZ^* = 0$, the following condition in abbreviated notation is obtained:

$$dZ^* = dZ + \mu_e d\psi + \nu_i d\alpha_i + \left[-Hdt + \lambda_i dy_i \right]_1^{2-}$$

$$+ \left[-Hdt + \lambda_i dy_i \right]_{4+}^{5-} + \left[-Hdt + \lambda_i dy_i \right]_{7+}^{8-} = 0 \; . \qquad (4.298)$$

This condition yields the following results

Point 1 dt: $\mu_1 + H_1 = 0$, (4.299)

du: $\mu_2 - \lambda_{u1} = 0$, (4.300)

dv: $\mu_3 - \lambda_{v1} = 0$, (4.301)

dx: $\mu_4 - \lambda_{x1} = 0$, (4.302)

dy: $\mu_5 - \lambda_{y1} = 0$, (4.303)

dm: $\mu_6 - \lambda_{m1} = 0$. (4.304)

Point 2 dt: $\mu_7 k_1 - \nu_2 g + \nu_3 u_2$

$$+ \nu_4 \; [v_2 - g(t_4 - t_2)] - \mu_{13} - H_{2-} = 0 \; , \qquad (4.305)$$

du: $-\nu_1 - \nu_3(t_4 - t_2) + \lambda_{u2-} = 0$, (4.306)

dv: $-\nu_2 - \nu_4(t_4 - t_2) + \lambda_{v2-} = 0$, (4.307)

dx: $-\nu_3 + \lambda_{x2-} = 0$, (4.308)

dy: $-\nu_4 + \lambda_{y2-} = 0$, (4.309)

dm: $-\mu_7 + \lambda_{m2-} = 0$. (4.310)

Point 4 dt: $-\mu_8 k_2 + \nu_2 g - \nu_3 u_2 + \nu_4 \; [-v_2 + g(t_4 - t_2)]$

$$+ \mu_{13} + H_{4+} = 0 \; , \qquad (4.311)$$

du: $\nu_1 - \lambda_{u4+} = 0$, (4.312)

dv: $\nu_2 - \lambda_{v4+} = 0$, (4.313)

dx: $\nu_3 - \lambda_{x4+} = 0$, (4.314)

dy: $\nu_4 - \lambda_{y4+} = 0$, (4.315)

dm: $\mu_7 - \lambda_{m4+} = 0$. (4.316)

Point 5 dt: $\mu_8 k_2 - \nu_6 g + \nu_7 u_5 + \nu_8 [v_5 - g(t_7 - t_5)]$

$- \mu_{14} - H_{5-} = 0$, (4.317)

du: $-\nu_5 - \nu_7(t_7 - t_5) + \lambda_{u5-} = 0$, (4.318)

dv: $-\nu_6 - \nu_8(t_7 - t_5) + \lambda_{v5-} = 0$, (4.319)

dx: $-\nu_7 + \lambda_{x5-} = 0$, (4.320)

dy: $-\nu_8 + \lambda_{y5-} = 0$, (4.321)

dm: $-\mu_8 + \lambda_{m5-} = 0$. (4.322)

Point 7: dt: $-\mu_9 k_3 + \nu_6 g - \nu_7 u_5 + \nu_8 [-v_5 + g(t_7 - t_5)]$

$- \mu_{14} + H_{7+} = 0$, (4.323)

du: $\nu_5 - \lambda_{u7+} = 0$, (4.324)

dv: $\nu_6 - \lambda_{v7+} = 0$, (4.325)

dx: $\nu_7 - \lambda_{x7+} = 0$, (4.236)

dy: $\nu_8 - \lambda_{y7+} = 0$, (4.327)

dm: $\mu_8 - \lambda_{m7+} = 0$. (4.328)

Point 8 dt: $\mu_9 k_3 - H_{8-} = 0$, (4.329)

du: $-\nu_9 + \lambda_{u8-} = 0$, (4.330) dv: $-\nu_{10} + \lambda_{v8-} = 0$, (4.331)

dx: $-\nu_{11} + \lambda_{x8-} = 0$, (4.332)

dy: $-\nu_{12} + \lambda_{y8-} = 0$, (4.333)

dm: $-\mu_9 + \lambda_{m8-} = 0$. (4.334)

Point f du: $\mu_{10} + \nu_9 = 0$, (4.335)

dv: $\mu_{11} + \nu_{10} = 0$, (4.336)

dx: $\nu_{11} = 0$, (4.337)

dy: $\mu_{12} + \nu_{12} = 0$, (4.338)

dm: $1 + \mu_9 = 0$. (4.339)

Equations (4.299) - (4.304) yield no usable information. Equations (4.305) - (4.316) may be paired off and combined with the other conditions to yield the following results:

$$H_{4+} = H_{2-} + \lambda_{m5-}k_2 - \lambda_{m2-}k_1 \ , \tag{4.340}$$

$$\lambda_{u4+} = \lambda_{u2-} - \lambda_{x2-}(t_4 - t_2) \ , \tag{4.341}$$

$$\lambda_{v4+} = \lambda_{v2-} - \lambda_{y2-}(t_4 - t_2) \ , \tag{4.342}$$

$$\lambda_{x4+} = \lambda_{x2-} \ , \tag{4.343}$$

$$\lambda_{y4+} = \lambda_{y2-} \ , \tag{4.344}$$

$$\lambda_{m4+} = \lambda_{m2-} \ . \tag{4.345}$$

Equations (4.317) - (4.328) may be paired off and combined with the other conditions to yield the following results:

$$H_{7+} = H_{5-} + \lambda_{m8-}k_3 + \lambda_{m5-}k_2 \ , \tag{4.346}$$

$$\lambda_{u7+} = \lambda_{u5-} - \lambda_{x5-}(t_7 - t_5) \ , \tag{4.347}$$

$$\lambda_{v7+} = \lambda_{v5-} - \lambda_{y5-}(t_7 - t_5) \ , \tag{4.348}$$

$$\lambda_{x7+} = \lambda_{x5-} \ , \tag{4.349}$$

$$\lambda_{y7+} = \lambda_{y5-} \ , \tag{4.350}$$

$$\lambda_{m7+} = \lambda_{m5-} \ . \tag{4.351}$$

Finally, the useful results obtained from the remaining equations (4.329) - (4.339) are as follows:

$$H_{8-} = \lambda_{m8-} k_3 \ , \tag{4.352}$$

$$\lambda_{x8-} = 0 \ , \tag{4.353}$$

$$\lambda_{m8-} = -1 \ . \tag{4.354}$$

Thus, $\lambda_x = 0$ throughout the trajectory. Since H is a constant along each of the unrestricted arcs, combining equations (4.352), (4.341) and (4.340) yields the following results:

$$L_1 \equiv H_{2-} - \lambda_{m2-} k_1 = 0 \tag{4.355}$$

$$L_2 \equiv H_{5-} - \lambda_{m5-} k_2 = 0 \tag{4.356}$$

$$L_3 \equiv H_{8-} - \lambda_{m8-} k_3 = 0 \tag{4.357}$$

These last three equations are identical to the results obtained by Mason and may be used as switching functions to determine t_2, t_5 and t_8 respectively.

Since the coasting time is fixed, $(t_4 - t_2)$ and $(t_7 - t_5)$, the equations of motion (4.258) - (4.262), Euler equations (4.264) - (4.269), End-point/corner restrictions (4.272) - (4.283), jump conditions (4.284) - (4.295), and finally the optimal endpoint/corner conditions (4.340) - (4.354) contain sufficient information to obtain a numerical solution. The procedure would be as follows:

1. Guesses are made for the unknown initial values of the Lagrange multipliers*.

2. The equations of motion plus Euler equations for the first stage are integrated until the condition $L_1 = 0$ is satisfied.

3. The jumps in the state variables and Lagrange multipliers are calculated.

4. The equations of motion plus the Euler equations are integrated for the second stage until the condition $L_2 = 0$ is satisfied.

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*Due to homogeneity one initial value may be arbitrarily fixed at some non-zero number. Then, equation (4.354) may be ignored.

5. The jumps in the state variables and Lagrange multipliers are calculated.

6. The equations of motion plus the Euler equations are integrated for the third stage until the condition $L_3 = 0$ is satisfied.

7. At this point a check is made to see if the final conditions

$$u_f = C$$

$$v_f = C$$

$$y_f = C$$

are satisfied. If not, steps 1-7 must be repeated until they are.

It is interesting to note that these results are easily reduced to the fixed structure case, i.e. when the mass discontinuities are fixed constants. For that case H is zero during each thrust period and non-zero during each coast. The switching functions are $L_1 \equiv H_{4+}$ and $L_2 \equiv H_{7+}$;a third switching function is unnecessary. Since L_1 determines t_2, the arguments of H_{4+} must be computed with the aid of equations (4.286) - (4.289) and (4.341) - (4.345), and similarly for L_2.

<u>Case II--thrust-coast-thrust optimal transfer problem</u> -In the calculation of optimal orbit transfer trajectories it is frequently necessary to consider the possibility that the trajectory is composed of intermediate coasting orbits. Recently de Veubeke[10] has examined this optimal transfer problem and has given an analytic solution with a coast period for inverse square force fields. By integrating the Euler equations along the coasting arc in closed form he was able to solve for the optimal transfer coast angle. The results of de Veubeke may also be obtained by using the methods of Section III which eliminate the necessity of integrating the Euler equations along the coasting arc.

Consider the problem of determining the optimal transfer of a constant thrust vehicle from one circular orbit to another coplaner circular orbit as shown in Figure (4.7). The objective is to maximize the payload during this process.

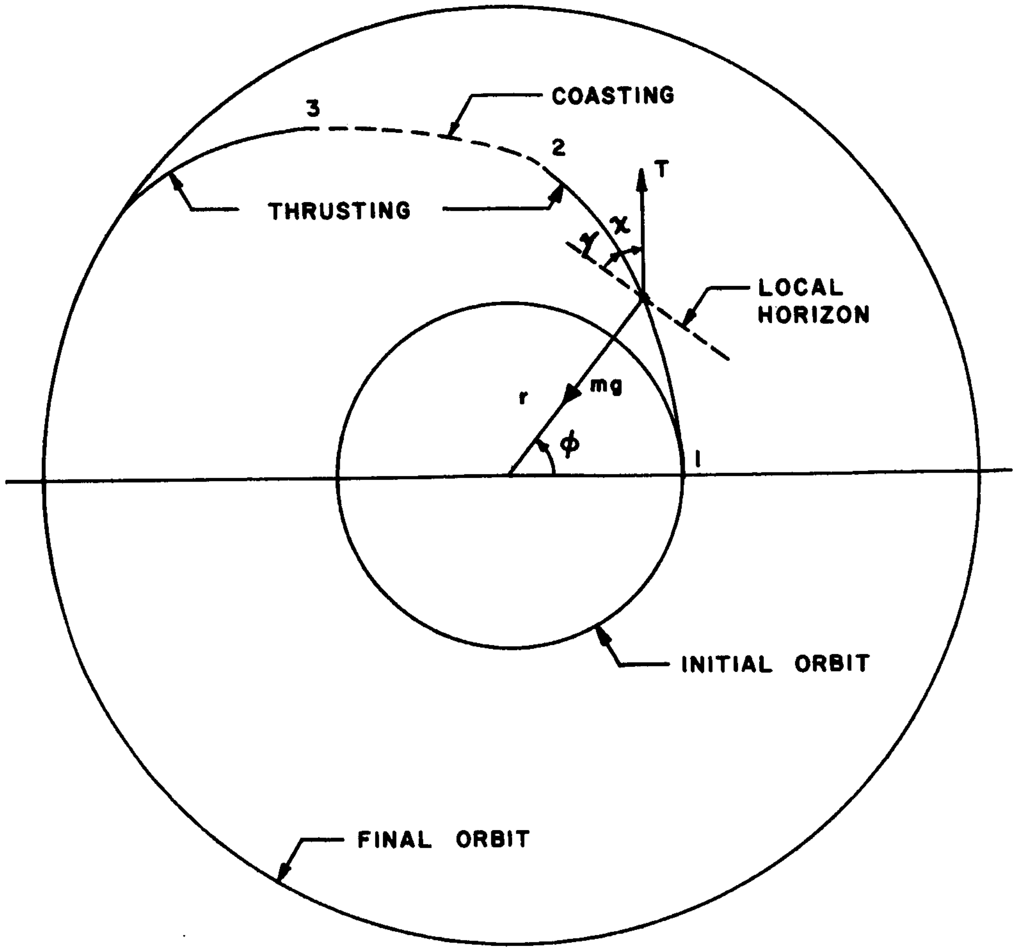

FIGURE 4.7 OPTIMAL ORBIT TRANSFER WITH A COASTING SEGMENT

The dynamical equations of constraint for the vehicle shown in Figure 4.7 are

$$\dot{v} = \frac{T}{m}\cos\chi - \frac{GM}{r^2}\sin\gamma\ , \tag{4.358}$$

$$\dot{\gamma} = \frac{T}{mv}\sin\chi - \frac{GM}{r^2 v}\cos\gamma + \frac{v}{r}\cos\gamma\ , \tag{4.359}$$

$$\dot{r} = v\sin\gamma\ , \tag{4.360}$$

$$\dot{\phi} = \frac{v}{r}\cos\gamma\ , \tag{4.361}$$

$$\dot{m} = -\beta\ , \tag{4.362}$$

where GM is the universal gravitational constant times the mass of the parent body. The constant thrust is given by T and m is the mass of the rocket. During thrusting periods the mass of the vehicle is assumed to decrease at a constant rate $-\beta$. The other symbols are defined in Figure 4.7. The state variables v, γ, r, ϕ, and m are subject to a single control variable χ which leaves one degree of freedom for optimal control.

Only one coasting period will be assumed, hence the trajectory will have one restricted arc between the points 2 and 3 as shown in Figure 4.7. Setting this problem up in the Mayer form ($Z = m_f$, $F = 0$) the Hamiltonian function becomes

$$H = \lambda_v\left(\frac{T}{m}\cos\chi - \frac{GM}{r^2}\sin\gamma\right) + \lambda_\gamma\left(\frac{T}{mv}\sin\chi - \frac{GM}{r^2 v}\cos\gamma + \frac{v}{r}\cos\gamma\right)$$

$$+ \lambda_r v\sin\gamma + \lambda_\phi\frac{v}{r}\cos\gamma - \lambda_m\beta\ . \qquad (4.363)$$

The Euler equations applicable for arcs 1-2 and 3-f are as follows:

v Euler Equation: $$\dot{\lambda}_v = \lambda_\gamma\left(\frac{T}{mv^2}\sin\chi - \frac{GM}{r^2v^2}\cos\gamma - \frac{\cos\gamma}{r}\right)$$

$$- \lambda_r\sin\gamma - \lambda_\phi\frac{\cos\gamma}{r}\ , \qquad (4.364)$$

γ Euler Equation: $$\dot{\lambda}_\gamma = \lambda_v\left(\frac{GM}{r^2}\cos\gamma\right) - \lambda_\gamma\left(\frac{GM}{r^2v} - \frac{v}{r}\right)\sin\gamma$$

$$- \lambda_r v\cos\gamma + \lambda_\phi\frac{v}{r}\sin\gamma\ , \qquad (4.365)$$

r Euler Equation: $$\dot{\lambda}_r = -2\lambda_v\frac{GM}{r^3}\sin\gamma - \lambda_\gamma\left(2\frac{GM}{r^3v} - \frac{v}{r^2}\cos\gamma\right)$$

$$+ \lambda_\phi\frac{v}{r^2}\cos\gamma\ , \qquad (4.366)$$

ϕ Euler Equation: $$\dot{\lambda}_\phi = 0\ , \qquad (4.367)$$

m Euler Equation: $$\dot{\lambda}_m = \lambda_v\frac{T}{m^2}\cos\chi + \lambda_\gamma\frac{T}{m^2v}\sin\chi\ , \qquad (4.368)$$

χ Euler Equation: $$\tan\chi = \frac{\lambda_\gamma}{v\lambda_v}\ . \qquad (4.369)$$

with the following first integral to the Euler equations

$$H = \text{constant}\ . \qquad (4.370)$$

Following the methods of Section III the arcs 1-2 and 3-f as shown in Figure 4.7 may now be eliminated. In this case

$$Z = m_f \ , \tag{4.371}$$

$$\beta = 0 \ . \tag{4.372}$$

The following conditions are applicable at the initial and final points of the trajectory.

$$\psi_1 = t_1 = 0 \ , \tag{4.373}$$

$$\psi_2 = v_1 - C = 0 \ , \tag{4.374}$$

$$\psi_3 = \gamma_1 = 0 \ , \tag{4.375}$$

$$\psi_4 = r_1 - C = 0 \ , \tag{4.376}$$

$$\psi_5 = \phi_1 = 0 \ , \tag{4.377}$$

$$\psi_6 = m_1 - C = 0 \ , \tag{4.378}$$

$$\psi_7 = v_f - C = 0 \ , \tag{4.379}$$

$$\psi_8 = \gamma_f = 0 \ , \tag{4.380}$$

$$\psi_9 = r_f - C = 0 \ . \tag{4.381}$$

Thus the final range angle ϕ and final time t, are left free. The "jump" conditions in the state variables between the points where coast begins (point 2) and the point where re-ignition takes place (point 3) may be obtained from the dynamical equations of constraint by setting T = 0 and $\beta = 0$. However, under this circumstance these equations reduce to those governing the well-known two-body problem. Thus the "jump" in v may be obtained from the energy equation

$$\alpha_1 = \frac{1}{2}{v_3}^2 - \frac{1}{2}{v_2}^2 - \frac{GM}{r_3} + \frac{GM}{r_2} = 0. \tag{4.382}$$

and the "jump" in γ may be obtained implicitly from the conservation of momentum

$$\alpha_2 = r_3 v_3 \cos\gamma_3 - r_2 v_2 \cos\gamma_2 = 0 \ . \tag{4.383}$$

The "jump" in r may be obtained from the equation of the path

$$r = \frac{h^2/GM}{1 + e\cos\phi} \qquad \begin{aligned} h &= h(r_2 v_2 \gamma_2) \\ e &= e(r_2 v_2 \gamma_2) \end{aligned} \tag{4.384}$$

However, for convenience of calculation, equation (4.384) evaluated at points 2 and 3 will be written as follows

$$\alpha_3 = \phi_3 - \phi_2 - G_3(r_2 r_3 v_2 \gamma_2) + G_2(r_2 v_2 \gamma_2) = 0 . \tag{4.385}$$

The "jump" in ϕ is obtained implicitly from the equation

$$r^2\dot{\phi} = h , \tag{4.386}$$

to give

$$\alpha_4 = -\int_{\phi_2}^{\phi_3} \frac{r^2}{h}\, d\phi + t_3 - t_2 = 0 . \tag{4.387}$$

Finally the "jump" in m is given by

$$\alpha_5 = m_3 - m_2 = 0 . \tag{4.388}$$

The θ function for this case is identically zero. By forming Z^* and setting $dZ^* = 0$, the following condition in abbreviated notation is obtained

$$dZ^* = dZ + \mu_e d\psi_e + \nu_i d\alpha_i + \Big[-Hdt + \lambda_i dy_i\Big]_1^{2-} + \Big[-Hdt + \lambda_i dy_i\Big]_{3+}^{f} = 0 . \tag{4.389}$$

This condition yields the following results:

Point 1 dt: $\mu_1 + H_1 = 0$, (4.390)

dv: $\mu_2 - \lambda_{v1} = 0$, (4.391)

dγ: $\mu_3 - \lambda_{\gamma 1} = 0$, (4.392)

dr: $\mu_4 - \lambda_{r1} = 0$, (4.393)

dϕ: $\mu_5 - \lambda_{\phi i} = 0$, (4.394) dm: $\mu_6 - \lambda_{m1} = 0$. (4.395)

Point 2 dt: $-\nu_4 - H_{2-} = 0$, (4.396)

dv: $-\nu_1 v_2 - \nu_2 r_2 \cos \gamma_2 + \nu_3 \dfrac{\partial\alpha_3}{\partial v_2} + \nu_4 \dfrac{\partial\alpha_4}{\partial v_2} + \lambda_{v2-} = 0$, (4.397)

dγ: $\nu_2 r_2 v_2 \sin \gamma_2 + \nu_3 \dfrac{\partial\alpha_3}{\partial\gamma_2} + \nu_4 \dfrac{\partial\alpha_4}{\partial\gamma_2} + \lambda_{\gamma 2-} = 0$, (4.398)

dr: $-\nu_1 \dfrac{GM}{r_2^2} - \nu_2 v_2 \cos \gamma_2 + \nu_3 \dfrac{\partial\alpha_3}{\partial r_2} + \nu_4 \dfrac{\partial\alpha_4}{\partial r_2} + \lambda_{r2-} = 0$, (4.399)

dϕ: $-\nu_3 + \nu_4 \dfrac{r_2^2}{h} + \lambda_{\phi 2-} = 0$, (4.400)

dm: $-\nu_5 + \lambda_{m2-} = 0$. (4.401)

Point 3 dt: $\nu_4 + H_{3+} = 0$, (4.402)

dv: $\nu_1 v_3 + \nu_2 r_3 \cos \gamma_3 - \lambda_{v3+} = 0$, (4.403)

dγ: $-\nu_2 r_3 v_3 \sin \gamma_3 - \lambda_{\gamma 3+} = 0$, (4.404)

dr: $\nu_1 \dfrac{GM}{r_3^2} + \nu_2 v_3 \cos \gamma_3 + \nu_3 \dfrac{\partial\alpha_3}{\partial r_3} - \lambda_{r3+} = 0$, (4.405)

dϕ: $\nu_3 - \nu_4 \dfrac{r_3^2}{h} - \lambda_{\phi 3+} = 0$, (4.406)

dm: $\nu_5 - \lambda_{m3+} = 0$. (4.407)

Point f dt: $-H_f = 0$, (4.408)

dv: $\mu_7 + \lambda_{vf} = 0$, (4.409)

dγ: $\mu_8 + \lambda_{\gamma f} = 0$, (4.410)

dr: $\mu_9 + \lambda_{\gamma f} = 0$, (4.411)

dϕ: $\lambda_{\phi f} = 0$, (4.412)

$$\text{dm:} \qquad 1 + \lambda_{mf} = 0 \ . \tag{4.413}$$

Equations (4.390) - (4.395) yield no usable information. Equation (4.396) combined with (4.402) yields

$$H_{3+} = H_{2-} \ . \tag{4.414}$$

This result combined with equation (4.408) and the fact that H is constant on the arcs 1-2 and 3-f shows that

$$H = 0 \tag{4.415}$$

on each of the unrestricted trajectories.

This result combined with equation (4.396) or (4.402) shows that

$$\nu_4 = 0 \ . \tag{4.416}$$

Thus when equation (4.400) is added to equation (4.406) the following result is obtained

$$\lambda_{\phi 2-} = \lambda_{\phi 3+} \ . \tag{4.417}$$

Since λ_ϕ is a constant along each of the unrestricted arcs [equation 4.367)] the condition given by equation (4.412) requires that

$$\lambda_\phi = 0 \ , \tag{4.418}$$

along arcs 1-2 and 3-f. Thus,either equation (4.400) or (4.406) may be used to show that

$$\nu_3 = 0 \ . \tag{4.419}$$

Multiplying equation (4.398) by cot γ_2 and equation (4.404) by cot γ_3 and adding [making use of equations (4.383), (4.416), and (4.419)] results in

$$\lambda_{\gamma 3+} = \lambda_{\gamma 2-} \frac{\tan \gamma_3}{\tan \gamma_2} \ . \tag{4.420}$$

Equation (4.398) and (4.404) may be solved for ν_2 to give

$$\nu_2 = - \frac{\cot \gamma_2}{r_2 v_2 \cos \gamma_2} \lambda_{\gamma 2-} = - \frac{\cot \gamma_3}{r_3 v_3 \cos \gamma_3} \lambda_{\gamma 3+} \ . \tag{4.421}$$

If equation (4.397) is divided by v_2 and equation (4.403) divided by v_3, then the resulting equations may be added and with the use of equation

(4.421) yield

$$\lambda_{v3+} = \lambda_{v2-}\left(\frac{v_3}{v_2}\right) + v_3 \left[\frac{\cot \gamma_2}{v_2^{\,2}} \lambda_{\gamma 2-} - \frac{\cot \gamma_3}{v_3^{\,2}} \lambda_{\gamma 3+}\right] . \tag{4.422}$$

The multiplier ν_1 may now be evaluated from equations (4.397) and (4.403) to give

$$\nu_1 = \frac{\lambda_{v2-}}{v_2} + \frac{\cot \gamma_2}{v_2^{\,2}} \lambda_{\gamma 2-} = \frac{\lambda_{v3+}}{v_3} + \frac{\cot \gamma_3}{v_3^{\,2}} \lambda_{\gamma 3+} . \tag{4.423}$$

Substituting for ν_1 and ν_2 from equations (4.421) and (4.423) into equations (4.399) and (4.405) yields:

$$-\frac{GM}{r_2^{\,2}} \sin \gamma_2 \lambda_{v2-} + \left(\frac{v_2}{r_2} - \frac{GM}{r_2^{\,2} v_2}\right) \cos \gamma_2 \lambda_{\gamma 2-} + v_2 \sin \gamma_2 \lambda_{r2-} = 0 , \tag{4.424}$$

and

$$-\frac{GM}{r_3^{\,2}} \sin \gamma_3 \lambda_{v3+} + \left(\frac{v_3}{r_3} - \frac{GM}{r_3^{\,2} v_3}\right) \cos \gamma_3 \lambda_{\gamma 3+} + v_3 \sin \gamma_3 \lambda_{r3+} = 0 . \tag{4.425}$$

Finally equations (4.401) and (4.407) may be combined to yield

$$\lambda_{m3+} = \lambda_{m2-} . \tag{4.426}$$

With a considerable amount of involved algebraic manipulations, the above equations may now be put into a more useful form. Assuming that $T_3 = T_2$, $\beta_3 = \beta_2$, and $m_3 = m_2$, and substituting equations (4.424), (4.425) and (4.426) into equation (4.414) yields

$$\left[\lambda_{v3+} \cos \chi_{3+} + \frac{\lambda_{\gamma 3+}}{v_3} \sin \chi_{3+}\right] = \left[\lambda_{v2-} \cos \chi_{2-} + \frac{\lambda_{\gamma 2-}}{v_2} \sin \chi_{2-}\right] . \tag{4.427}$$

From the control equation

$$\sin \chi = \frac{\lambda_\gamma}{v\lambda} , \text{ and } \cos \chi = \frac{\lambda_v}{\lambda} , \tag{4.428}$$

where $\lambda^2 = \lambda_v^{\,2} + \left(\frac{\lambda_\gamma}{v}\right)^2$. Therefore equation (4.427) becomes

$$\lambda_{3+} = \lambda_{2-} . \tag{4.429}$$

Combining equations (4.422) and (4.420) gives

$$\lambda_{v3+}^{2} = v_3^{2}\left[\frac{\lambda_{v2-}^{2}}{v_2^{2}} + 2\left(\frac{1}{v_2^{2}} - \frac{1}{v_3^{2}}\right)\frac{\cot\gamma_2\lambda_{\gamma 2-}\lambda_{v2-}}{v_2} + \left(\frac{1}{v_2^{2}} - \frac{1}{v_3^{2}}\right)^{2}\cot^{2}\gamma_2\lambda_{\gamma 2-}^{2}\right] . \quad (4.430)$$

From equation (4.383)

$$\cos\gamma_3 = \frac{r_2 v_2 \cos\gamma_2}{r_3 v_3} = \frac{\cos\gamma_2}{RV} , \quad (4.431)$$

where $R = \frac{r_3}{r_2}$ and $V = \frac{v_3}{v_2}$. Thus

$$\tan^{2}\gamma_3 = \frac{R^2 V^2}{\cos^{2}\gamma_2} - 1 , \quad (4.432)$$

and from equation (4.420)

$$\frac{\lambda_{\gamma 3+}^{2}}{v_3^{2}} = \frac{\lambda_{\gamma 2-}^{2}\cot^{2}\gamma_{2-}}{v_2^{2}V^2}\left(\frac{R^2V^2}{\cos^{2}\gamma_2} - 1\right) . \quad (4.433)$$

Substituting equations (4.430) and (4.433) into (4.427) yields,

$$v_3^{2}\left[\frac{\lambda v_{2-}^{2}}{v_2^{2}} + 2\left(\frac{1}{v_2^{2}} - \frac{1}{v_3^{2}}\right)\frac{\cot\gamma_2\lambda_{\gamma 2-}\lambda_{v2-}}{v_2} + \left(\frac{1}{v_2^{2}} - \frac{1}{v_3^{2}}\right)^{2}\cot^{2}\gamma_2\lambda_{\gamma 2-}^{2}\right]$$

$$+ \frac{\lambda\gamma_{2-}^{2}\cot\gamma_2}{v_2^{2}V^2}\left[\frac{R^2V^2}{\cos^{2}\gamma_2} - 1\right] = \lambda_{v2-}^{2} + \frac{\lambda_{\gamma 2-}^{2}}{v_2^{2}} . \quad (4.434)$$

Dividing by $\lambda^2_{\gamma 2-}$ and replacing $\dfrac{v_2 \lambda_{v2-}}{\lambda_{\gamma 2-}}$ by $\cot \chi_2$ gives,

$$v_3^2 \left[\frac{\cot^2 \chi_2}{v_2^4} + 2\left(\frac{1}{v_2^2} - \frac{1}{v_3^2}\right) \frac{\cot \gamma_2 \cot \chi_2}{v_2^2} + \left(\frac{1}{v_2^2} - \frac{1}{v_3^2}\right) \cot^2 \gamma_2 \right]$$

$$+ \frac{\cot^2 \gamma_2}{v_2^2} \left[\frac{R^2}{\cos^2 \gamma_2} - \frac{1}{V^2} \right] = \frac{\cot^2 \chi_2}{v_2^2} + \frac{1}{v_2^2} \ . \tag{4.435}$$

Multiplying by v_2^2 and rearranging terms yields,

$$(V^2 - 1) \cot^2 \chi_2 + 2(V^2 - 1) \cot \gamma_2 \cot \chi_2 + \frac{(V^2 - 1)^2}{V^2} \cot^2 \gamma_2$$

$$= 1 - \left[\frac{R^2}{\cos^2 \gamma_2} - \frac{1}{V^2} \right] \cot^2 \gamma_2 \ , \tag{4.436}$$

or simply

$$(V^2 - 1) \left[\cot^2 \chi_2 + 2 \cot \gamma_2 \cot \chi_2 + \cot^2 \gamma_2 \right] = \frac{1 - R^2}{\sin^2 \gamma_2} \ . \tag{4.437}$$

but

$$V^2 - 1 = \frac{2GM}{r_2 v_2^2} \left[\frac{1 - R}{R} \right] . \tag{4.438}$$

Thus, by letting

$$K = \frac{2GM \sin^2 \gamma_2}{r_2 v_2^2} \left[\cot \gamma_2 + \cot \chi_2 \right]^2 \ , \tag{4.439}$$

The following result is finally obtained from equation (4.437)

$$(1 - R)\ (R^2 + R - K) = 0 \ . \tag{4.440}$$

The roots for this equation are

$$R_1 = 1 \tag{4.441}$$

$$R_2 = \frac{\sqrt{1 + 4K} - 1}{2} \tag{4.442}$$

$$R_3 = - \left[\frac{\sqrt{1 + 4K} + 1}{2} \right] \tag{4.443}$$

Since $R > 0$ the third root must be discarded. This leaves two possible solutions which are equivalent to those given by de Veubeke.

The solutions given by equations (4.441) and (4.442) can each be interpreted in two different ways. The root R_1 implies either

(a) No coast (Degenerate solution)
or (b) Coast to a symmetric point (Symmetric solution)

The root R_2 implies either a coasting arc to a point

(c) on the same side of the major axis as point 2 (Asymmetric solution)

or a coasting arc to a point

(d) on the opposite side of the major axis as point 2 (Symmetric-Asymmetric solution)

If the coasting trajectory at point 2 is an ellipse, all solutions are physically realizable (provided that $r_{perigee} \leq r_3 \leq r_{apogee}$). If the coasting arc at point 2 is a parabola or a hyperbola, then the physically realizable solutions depend on $\dot{r}_2$ and R_2 as follows

(a) If $\dot{r}_2 < 0$ then all solutions are possible except for the asymmetric one with $R_2 > 1$.
(b) If $\dot{r}_2 > 0$ then only the degenerate solution and the asymmetric solution with $R_2 > 1$ are possible.

Using the information developed so far, the following procedure may now be used to obtain a solution:

1. Guesses are made for the unknown initial values of the Lagrange multipliers.

2. The equations of motion plus the Euler equations for the first thrusting arc are integrated until the condition given by equation (4.424) is satisfied.

3. A choice must now be made for R among the possible physically realizable solutions and a "jump" made to the point 3 using equations (4.382) - (4.388) and equations (4.417), (4.420), (4.422), (4.425) and (4.426).

4. The equations of motion plus the Euler equations for the final thrusting arc are then integrated and a check is made to see if the final endpoint is intercepted.

The conditions developed in this paper are not sufficient to dictate the proper choice for R at the point 2. It is apparent that there is need for further study into this matter.

SECTION V

DISCUSSION AND CONCLUSIONS

The space mechanics and flight mechanics problems presented in Section IV possess the following characteristics:

1. They are subject to a single control variable.
2. The control variable can be extremized only over certain portions of the total trajectory. Over these portions, the problem falls under the clarification of the "Problem of Bolza".
3. Over the remaining portions of the trajectory the control is prescribed.

The theory presented in Sections II and III is developed for problems of the above type. It is based on the requirement that the dynamical constraint equations are integrable analytically along those arcs over which the control is prescribed. Under such circumstances, the theory yields a solution by following two distinct procedures.

1. The Euler equations are solved only over the unrestricted portions of the trajectory to determine the optimal control.
2. A set of "corner conditions" containing relations involving endpoints of the restricted segments are then solved in conjunction with the Euler equations to yield the optimal endpoints and corner points. The relations involving the endpoints of the restricted segments are determined by integrating the dynamical constraint equations along the restricted arcs.

The following conclusions are made on the basis of applying the above procedure to the problems presented in Section IV.

1. Analysis and computation for a given problem are simplified by eliminating the need to integrate augmented Euler equations along the restricted arcs.
2. The theory is relatively simple to use and is universally applicable to all problems with restricted arcs independent of how the arc is generated.
3. The requirement of being able to integrate the restricted equations of motion does not present any particular difficulty for the large class of space mechanics problems which involve coasting periods. (Assuming that the coast period may be treated as a classical two body problem)
4. There are several other important problems in which the integration requirement does not present difficulties.
5. For certain problems, such as the optimal staging with coast periods one, switching functions are obtained which locate the beginning of the restricted arc. These functions depend on state variables evaluated at the end of the restricted arc. This situation would complicate any procedure requiring integration of the

Euler equations, whereas it does not complicate the arc elimination method used here.

6. Parameters may be introduced to make problems with discontinuities continuous again; however, the use of parameters does not necessarily simplify computation needed for a solution.

Neither the theory nor the extent of the applications presented in this paper may be regarded as being complete. As was pointed out at the end of Section IV, the "corner conditions" developed here are necessary but not sufficient conditions for the location of a corner point. There is a need for the development of sufficiency conditions for the type of problems considered here that would dictate the proper solution for the location of a corner when more than one is obtained.

Problems which do not satisfy the integration requirement may be handled in some cases by using "engineering approximations". For example, if coasting periods are short an assumption of constant gravity magnitude and direction is frequently appropriate. Also, for low-thrust rocket vehicles maneuvering in the vicinity of a planet the thrust force is negligible in comparison with the gravitational force. Thus, one may conclude that by proper use of engineering assumptions and physical interpretation it is possible to apply the arc elimination technique to a much larger class of problems.

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